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STALLINGS' FOLDINGS AND SUBGROUPS OF AMALGAMS OF FINITE GROUPS

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Abstract. In the 1980's Stallings [35] showed that every finitely generated subgroup of a free group is canonically represented by a finite minimal immersion of a bouquet of circles. In terms of the theory of automata, this is a minimal finite inverse automaton. This allows for the deep algorithmic theory of finite automata and finite inverse monoids to be used to answer questions about finitely generated subgroups of free groups.

In this paper we attempt to apply the same methods to other classes of groups. A fundamental new problem is that the Stallings folding algorithm must be modified to allow for "sewing" on relations of non-free groups. We look at the class of groups that are amalgams of finite groups. It is known that these groups are locally quasiconvex and thus all finitely generated subgroups are represented by finite automata. We present an algorithm to compute such a finite automaton and use it to solve various algorithmic problems.

1. Introduction

This paper has as its main goal the extension of results from the case of studying subgroups of free groups to that of other classes of finitely presented groups via techniques in automata theory and the theory of inverse semigroups. The main idea is that a finitely generated subgroup H of a "nice" group G can be represented by a finite directed graph labelled by generators of G. From an automata theoretical point of view, the graph is a finite inverse automaton, and from a topological point of view it is an immersion over a bouquet of circles. This convergence of ideas from group theory, topology, and the theory of finite automata and finite semigroups allows for a rich interaction of ideas and methods from many different fields.

In the free group, finitely generated subgroups correspond precisely to finite inverse automata or equivalently to finite immersions over a bouquet of circles. This object can be constructed algorithmically by the process of Stallings foldings [35]. It can be shown that every finitely generated subgroup H of a free group $FG(X)$ over a set of generators X corresponds to a uniquely determined such finite object $\mathcal{A}(H)$ which is in fact a topological

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invariant of H. Another important invariant of H is its syntactic monoid $\mathcal{M}(H)$ which is the transition monoid of $\mathcal{A}(H)$. This is a finite inverse monoid. Thus combinatorial and algorithmic properties of H can be studied by looking at the finite objects $\mathcal{A}(H)$ and $\mathcal{M}(H)$. Since the theory of finite automata and finite semigroups have rich algorithmic theories, non-trivial results can be obtained in this way. In particular, this approach gives polynomial time algorithms to solve the membership problem for H (i.e. the Generalized Word Problem), the finite index problem, and the computation of closures of H in various profinite topologies. On the other hand, the problem of checking purity, that is, if H is closed under taking roots, turns out to be PSPACE-complete. See the articles [2, 5, 17, 25, 26] for these and other examples of this approach.

In general, the results mentioned above can not be extended to every class of groups. That is because this immediately runs into a problem: a theorem of Mikhailova [23] shows that the membership problem for the direct product of two non-Abelian free groups is undecidable. Thus any hope of generalizing these results to other classes of groups first must choose a class of groups that are in some sense close to free groups, but far from the direct product of two free groups!

The groups considered in this paper are *amalgams of finite groups*. As is well known, such groups are *hyperbolic* ([3]) and *locally quasiconvex* ([16]). The combination of these properties provides a fulfillment of the above requirement.

Recall, that a group G is *locally quasiconvex* if and only if every finitely generated subgroup H of G is quasiconvex. In general, quasiconvexity of the subgroup H depends on the presentation of the group G. However if the group G is also *hyperbolic*, then the subgroup H remains quasiconvex in all finite presentations of G ([12]). This enables us to work with a fixed finite presentation of G without loss of generality.

In [10] Gitik proved that the subgroup H of the group G is quasiconvex if and only if the *geodesic core* $Core(G,H)$ of $Cayley(G,H)$ (which is the union of all closed geodesics in the relative Cayley graph $Cayley(G,H)$ beginning at the basepoint $H \cdot 1$) is finite. Thus local quasiconvexity of the group G (with a fixed finite presentation) ensures the existence of a finite graph canonically associated with the given subgroup H. Such a graph posses all the essential information about the subgroup H itself, therefore it can be used to study properties of H.

However the geodesic core can not be constructed using a generalization of Stallings' foldings algorithm. That is because in amalgams, unlike in free groups, the theoretically well defined notion of geodesic is ambiguous from computational and constructible points of view. We are not familiar with any rewriting procedure that computes geodesic words in amalgams. Since Stallings' foldings can be viewed as a simulation of a rewriting procedure of

freely reduced words in free groups, these methods do not appear useful for a construction of geodesic cores.

In spite of this, normal (reduced) words do have a good realization in amalgams given by their standard group presentation. Indeed, there is a well known rewriting procedure ([23]) that given an element of an amalgam, computes its normal (reduced) form. Such a rewriting is possible when the amalgamated subgroup has a solvable membership problem in the factors. Therefore it can be applied to elements of amalgams of finite groups. This allows us to generalize Stallings' algorithm following similar ideas and techniques.

Moreover, the following lemma of Gitik shows that geodesics and strong normal paths are close to each other, which ensures in our case the finiteness of the *normal core* of $Cayley(G, H)$, that is the union of all closed normal paths in $Cayley(G, H)$ starting at the basepoint $H \cdot 1$

Lemma 1.1 (Lemma 4.1 in [10])**.** *If G_1 and G_2 are quasiconvex subgroups of a hyperbolic group $G = G_1 \ast_A G_2$, then there exists a constant $\epsilon \geq 0$ such that for any geodesic $\gamma \subset Cayley(G)$ there exists a path γ' in normal form with the same endpoints as γ with the following properties:*

(1) *$\gamma \subset N_\epsilon(\gamma')$ [1] and $\gamma' \subset N_\epsilon(\gamma)$,*
(2) *an endpoint of any maximal monochromatic subpath of γ' lies in γ and is bichromatic in γ.*

We explore normal cores and find that they can be defined not only theoretically, but constructively as well. Theorem 7.5 says that the normal core of $Cayley(G, H)$ is a *reduced precover* of G (see Definition 6.18), which is a restriction of the notion of *precovers*(see Section 6) presented by Gitik in [11]. Roughly speaking, one can think of a reduced precover as a bunch of "essential" copies of relative Cayley graphs of the free factors of G glued to each other according to the amalgamation. We prove (Corollary 7.4) that reduced precovers determining the same subgroup are isomorphic. Furthermore, our Maim Theorem (Theorem 7.1) states that given a finitely generated subgroup H of an amalgam $G = G_1 \ast_A G_2$ there exists a unique reduced precover determining H, which is the normal core of $Cayley(G, H)$.

This constructive characterization of normal cores enables us to present a quadratic algorithm (see Section 8) that given a finite set of subgroup generators of H constructs the normal core of $Cayley(G, H)$, where G is an amalgam of finite groups. Theorem 8.9 provides the validity and the finiteness of the construction.

Thus the normal core Δ of $Cayley(G, H)$ posses properties analogous to those of graphs constructed by the Stallings' algorithm for finitely generated subgroups of free groups. Geometrically, it can be viewed as the 1-skeleton of a topological core of the covering space corresponding to H of the standard

[1] $N_K(S) = \cup\{p \,|\, p$ is a geodesic in $Cayley(G)$, $\iota(p) \in S$, $|p| \leq K\}$ is the K*-neighborhood* of S.

2-complex of G. Algebraically, Δ is an "essential part" of the relative Cayley graph $Cayley(G, H)$, and from the automata theoretic point of view, it is a minimal finite inverse automaton $\mathcal{A}$ such that $L(\mathcal{A}) =_G H$.

Furthermore, Theorem 8.9 ensures the canonicity of our construction, that is its independence from the choice of subgroup generators, and guarantees that the resulting graph $\Gamma(H)$ "accepts" all normal forms of elements from H. We get the following corollary which gives an immediate solution for the membership problem of H.

Corollary 1.2. *A normal word g is in H if and only if it labels a closed path from v_0 to itself in $\Gamma(H)$.*

An application of normal cores yields polynomial (mostly quadratic) solutions for a nice list of algorithmic problems concerning finitely generated subgroups of amalgams of finite groups: the membership problem, the finite index problem, the freeness problem, the power problem, the conjugacy problem, the normality and the malnormality problems. Furthermore, the separability problem can be solved in some particular cases and an effective Kurosh decomposition for finitely generated subgroups in the case of free products can be found. All these results are presented in the PhD thesis of the author [27].

The present paper includes only the solution for the membership problem as a demonstration of the effectiveness of our methods. The rest of the above algorithmic problems and their solutions will appear in our future papers [28, 29].

Finally, we notice that there are several generalization results of Stallings' algorithm to other classes of groups. Schupp in [32] presents an algorithm for certain Coxeter groups and surface groups of an extra-large type. Kapovich and Schupp [19] make use of modified Stallings' foldings and the minimization technique of Arzhantseva and Ol'shanskii [2] to present finitely generated subgroups of Coxeter groups and Artin groups of extra-large type and also of one-relator groups with torsion by labelled graphs. Kapovich, Weidman, and Miasnikov in [18] develop a combinatorial treatment of Stallings' foldings in the context of graphs of groups through the use of the Bass-Serre theory. McCammond and Wise [30] generalize Stallings' algorithm for the class of coherence groups, however the resulting graphs are not canonical (they depend on the choice of subgroup generators). Hence they are not suitable for solving algorithmic problems for subgroups via their graphs. Recently Miasnikov, Remeslennikov and Serbin have generalized Stallings' algorithm to the class of fully residually free groups [31]. The developed methods were applied to solve a collection of algorithmic problems concerning this class of groups in [20].

Other Methods. There have been a number of papers, where methods, not based on Stallings' foldings, have been presented. One can use these methods to treat finitely generated subgroups of amalgams of finite groups.

A topological approach can be found in works of Bogopolskii [6, 7]. For the automata theoretic approach, see papers of Holt and Hurt [14, 15], papers of Cremanns, Kuhn, Madlener and Otto [8, 21], as well as the recent paper of Lohrey and Senizergues [22].

However the methods for treating finitely generated subgroups presented in the above papers were applied to some particular subgroup property. No one of these papers have as its goal a solution of various algorithmic problems, which we consider as our primary aim. We view the current paper as the first step in its achieving. Similarly to the case of free groups (see [17]), our combinatorial approach seems to be the most natural one for this purpose. It yields reach algorithmic results, as appear in our future papers [28, 29].

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3. Labelled Graphs

The primary idea of this paper is to study finitely generated subgroups of amalgams of finite groups by constructing subgroup graphs exactly as in the case of free group. Hence we begin by fixing the notation on graphs that will be used along this work. In doing so we follow the notations used by Stallings in [35] and Gitik in [11].

At the end of the section we recall the notion of Stallings' foldings and introduce a new graph operation which is an immediate generalization of foldings for a non free group G. We prove that both operations when applied to a subgroup graph $\Gamma(H)$, $H \leq G$, do not affect the subgroup H.

A graph Γ consists of two sets $E(\Gamma)$ and $V(\Gamma)$, and two functions $E(\Gamma) \to E(\Gamma)$ and $E(\Gamma) \to V(\Gamma)$: for each $e \in E$ there is an element $\overline{e} \in E(\Gamma)$ and an element $\iota(e) \in V(\Gamma)$, such that $\overline{\overline{e}} = e$ and $\overline{e} \neq e$. The elements of $E(\Gamma)$ are called *edges*, and an $e \in E(\Gamma)$ is a *directed edge* of Γ, $\overline{e}$ is the *reverse (inverse) edge* of e. The elements of $V(\Gamma)$ are called *vertices*, $\iota(e)$ is the *initial vertex* of e, and $\tau(e) = \iota(\overline{e})$ is the *terminal vertex* of e. We call them the *endpoints* of the edge e.

Remark 3.1. A *subgraph* of Γ is a graph C such that $V(C) \subseteq V(\Gamma)$ and $E(C) \subseteq E(\Gamma)$. In this case, by abuse of language, we write $C \subseteq \Gamma$.

Similarly, whenever we write $\Gamma_1 \cup \Gamma_2$ or $\Gamma_1 \cap \Gamma_2$ we always mean that the set operations are, in fact, applied to the vertex sets and the edge sets of the corresponding graphs.

$\diamond$

A *labelling* of Γ by the set $X^{\pm}$ is a function

$$lab: \ E(\Gamma) \to X^{\pm}$$

such that for each $e \in E(\Gamma)$, $lab(\overline{e}) = (lab(e))^{-1}$.

The last equality enables one, when representing the labelled graph Γ as a directed diagram, to represent only X-labelled edges, because X^{-1}-labelled edges can be deduced immediately from them.

A graph with a labelling function is called a *labelled (with $X^{\pm}$) graph.* A labelled graph is called *well-labelled* if

$$\iota(e_1) = \iota(e_2), \ lab(e_1) = lab(e_2) \ \Rightarrow \ e_1 = e_2,$$

for each pair of edges $e_1, e_2 \in E(\Gamma)$. See Figure 1.

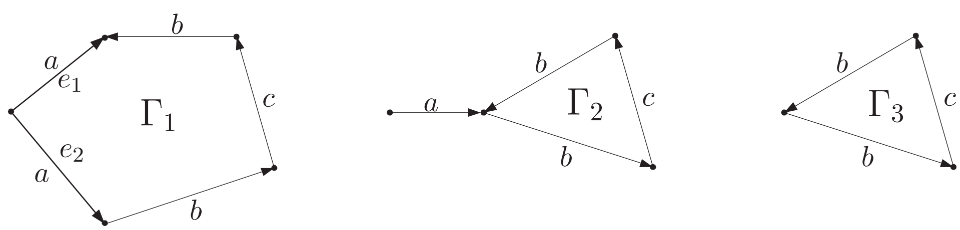

FIGURE 1. The graph Γ_1 is labelled with $\{a, b, c\}^{\pm}$, but it is not well-labelled. The graphs Γ_2 and Γ_3 are well-labelled with $\{a, b, c\}^{\pm}$.

The label of a path $p = e_1 e_2 \cdots e_n$ in Γ, where $e_i \in E(\Gamma)$, is the word

$$lab(p) \equiv lab(e_1) \cdots lab(e_n) \in (X^{\pm})^*.$$

Notice that the label of the empty path is the empty word. As usual, we identify the word $lab(p)$ with the corresponding element in $G = gp\langle X \mid R \rangle$.

Note that elements of $G = gp\langle X|R\rangle$ are equivalence classes of words. However it is customary to blur the distinction between a word u and the equivalence class containing u. We will distinguish between them by using different equality signs: $\boxed{\text{“}\equiv\text{”}}$ [2] for the equality of two words and $\boxed{\text{“}=_G\text{”}}$ to denote the equality of two elements of G, that is the equality of two equivalence classes.

A path $p = e_1 e_2 \cdots e_n$ is freely reduced if $e_{i+1} \neq \overline{e_i}$ for all $1 \leq i \leq n-1$.

Remark 3.2. If Γ is a well-labelled graph then a path p in Γ is freely reduced if and only if $lab(p)$ is a freely reduced word. $\diamond$

Denote the pair consisting of the graph Γ and the *basepoint* (a distinguished vertex of the graph Γ) v_0 by (Γ, v_0) and call it a *pointed graph.*

Following the notation of Gitik, [11], we denote the set of all closed paths in Γ starting at v_0 by $\boxed{Loop(\Gamma, v_0)}$, and the image of $lab(Loop(\Gamma, v_0))$ in G

[2]Throughout the present paper boxes are used to emphasize the notation.

by $\boxed{Lab(\Gamma, v_0)}$. More precisely,

$$Loop(\Gamma, v_0) = \{p \mid p \text{ is a path in } \Gamma \text{ with } \iota(p) = \tau(p) = v_0\},$$
$$Lab(\Gamma, v_0) = \{g \in G \mid \exists p \in Loop(\Gamma, v_0) \ : \ lab(p) =_G g\}.$$

Remark 3.3 ([11])**.** It is easy to see that $Lab(\Gamma, v_0)$ is a subgroup of G. $\diamond$

Remark 3.4. If $V(\Gamma) = \{v_0\}$ and $E(\Gamma) = \emptyset$ then we assume that $Lab(\Gamma, v_0) = \{1\}$. $\diamond$

Remark 3.5. We say that $H = Lab(\Gamma, v_0)$ is the subgroup of G *determined* by the graph Γ. Thus any pointed graph labelled by $X^{\pm}$, where X is a generating set of the group G, determines a subgroup of G. This argues the use of the name *subgroup graphs* for such graphs. $\diamond$

As is well known [5, 25, 17], well-labelled graphs, presented above combinatorially, can be viewed as algebraical, topological, geometrical and automata-theoretical objects as well. The detailed exploration of various connections between combinatorial group theory, semigroup theory and formal language theory can be found in [9].

Thus a finite pointed graph (Γ, v_0) well-labelled with $X^{\pm}$ can be viewed as the inverse automaton with the same initial-terminal state v_0:

$$\mathcal{A} = (V(\Gamma), X^{\pm}, \delta, v_0, \{v_0\}),$$

where $\delta : V(\Gamma) \times X^{\pm} \to V(\Gamma)$, usually denoted $\delta(v, x) = v \cdot x$, satisfies $\delta(v, x) = w$ if and only if there exist $e \in E(\Gamma)$ such that $\iota(e) = v$, $\tau(e) = w$ and $lab(e) \equiv x$. The representation of (Γ, v_0) is the positive state graph of $\mathcal{A}$ and $L(\mathcal{A}) = lab(Loop(\Gamma, v_0))$. The reader is referred to [5] for the missing definitions.

As usual, δ is extended to a (partial) function on $V(\Gamma) \times (X^{\pm})^*$ by letting $v \cdot 1 = v$ and $v \cdot (ua) = (v \cdot u) \cdot a$ (if this is defined) for all $v \in V(\Gamma)$, $u \in (X^{\pm})^*$ and $a \in X^{\pm}$. Thus if $v, w \in V(\Gamma)$ and p is a path in Γ such that

$$\iota(p) = v, \ \tau(p) = w \text{ and } lab(p) \equiv u,$$

then, following the automata theoretic notation, we simply write $v \cdot u = w$ to summarize this situation.

By abuse of language, we say that a word w is *accepted* by the graph (Γ, v_0) if and only if there exists a path p in Γ closed at v_0, $\iota(p) = \tau(p) = v_0$ such that $lab(p) \equiv w$, that is $v_0 \cdot w = v_0$.

Morphisms of Labelled Graphs. Let Γ and Δ be graphs labelled with $X^{\pm}$. The map $\pi : \Gamma \to \Delta$ is called a *morphism of labelled graphs*, if π takes vertices to vertices, edges to edges, preserves labels of directed edges and has the property that

$$\iota(\pi(e)) = \pi(\iota(e)) \text{ and } \tau(\pi(e)) = \pi(\tau(e)), \ \forall e \in E(\Gamma).$$

An injective morphism of labelled graphs is called an *embedding*. If π is an embedding then we say that the graph Γ *embeds* in the graph Δ.

A *morphism of pointed labelled graphs* $\pi : (\Gamma_1, v_1) \to (\Gamma_2, v_2)$ is a morphism of underlying labelled graphs $\pi : \Gamma_1 \to \Gamma_2$ which preserves the basepoint $\pi(v_1) = v_2$. If Γ_2 is well-labelled then there exists at most one such morphism ([17]).

Remark 3.6 ([17])**.** If two pointed well-labelled (with $X^{\pm}$) graphs (Γ_1, v_1) and (Γ_2, v_2) are isomorphic, then there exists a unique isomorphism $\pi : (\Gamma_1, v_1) \to (\Gamma_2, v_2)$. Therefore (Γ_1, v_1) and (Γ_2, v_2) can be identified via π. In this case we sometimes write $(\Gamma_1, v_1) = (\Gamma_2, v_2)$. $\diamond$

Graph Operations. Recall that a *Stallings' folding* is an identification of a pair of distinct edges with the same initial vertex and the same label. The operation of *"cutting hairs"* consists of removing from the graph edges whose terminal vertex has degree 1 (see Figure 1: the graph Γ_2 is obtained from the graph Γ_1 by folding the edges e_1 and e_2; the graph Γ_3 is obtained from the graph Γ_2 by cutting the hair edge labelled by a). As is well known [35, 25, 17], these graph operations don't affect the corresponding subgroup of a free group. The following lemma demonstrates the similar behavior in the case of finitely presented non free groups.

Lemma 3.7. *Let $G = gp\langle X|R\rangle$ be a finitely presented group. Let Γ be a graph labelled with $X^{\pm}$ and let Γ' be a graph labelled with $X^{\pm}$ obtained from Γ by a single folding or by "cutting" a single hair. Then $Lab(\Gamma, v_0) = Lab(\Gamma', v_0')$, where v_0 is the basepoint of Γ and v_0' is the corresponding basepoint of Γ'.*

Proof. Let $F(X)$ be a free group with finite free basis X. Let

$$lab : \ E(\Gamma) \to X^{\pm}$$

be the labelling function of Γ. The function *lab* extends to the labelling of paths of Γ such that the label of a path $p = e_1 e_2 \cdots e_n$ in Γ, is the word $lab(p) \equiv lab(e_1) \cdots lab(e_n) \in (X^{\pm})^*$. Denote by $Lab_{F(X)}(\Gamma)$ the image of $lab(Loop(\Gamma))$ in $F(X)$.

As is well known, [35, 25, 17], foldings and cutting hairs don't affect the fundamental group of the graph, i.e.

$$Lab_{F(X)}(\Gamma, v_0) = Lab_{F(X)}(\Gamma', v_0').$$

Since the homomorphism $(X^{\pm})^* \to G$ factors through $F(X)$

$$(X^{\pm})^* \to F(X) \to G,$$

we conclude that $Lab(\Gamma, v_0) = Lab(\Gamma', v_0')$.

$\diamond$

Let f_1 and f_2 be a pair of folded edges of the graph Γ with labels x and x^{-1}, respectively. Hence the path $\overline{f_1} f_2$ in Γ is labelled by the trivial relator $x^{-1}x$. The folding operation applied to the edges f_1 and f_2 implies the identification of the endpoints of $\overline{f_1} f_2$. Thus the natural extension of

such operation to the case of a non free group G is an identification of the endpoints of paths labelled by a relator.

Definition 3.8. Let Γ be a graph labelled with $X^{\pm}$. Suppose that p is a path of Γ with

$$v_1 = \iota(p) \neq \tau(p) = v_2 \text{ and } lab(p) =_G 1.$$

Let Δ be a graph labelled with $X^{\pm}$ defined as follows.

The vertex set of Δ is a vertex set of Γ with $\iota(p)$ and $\tau(p)$ removed and a new vertex ϑ added (we think of the vertices $\iota(p)$ and $\tau(p)$ as being identified to produce vertex ϑ):

$$V(\Delta) = (V(\Gamma) \setminus \{\iota(p), \tau(p)\}) \cup \{\vartheta\}.$$

The edge set of Δ is the edge set of Γ:

$$E(\Delta) = E(\Gamma).$$

The endpoints and arrows for the edges of Δ are defined in a natural way. Namely, if $e \in E(\Delta)$ and $\iota(e), \tau(e) \notin \{v_1, v_2\}$ then we put $\iota_\Delta(e) = \iota_\Gamma(e)$. Otherwise $\iota_\Delta(e) = \vartheta$ if $\iota_\Gamma(e) \in \{v_1, v_2\}$ and $\tau_\Delta(e) = \vartheta$ if $\tau_\Gamma(e) \in \{v_1, v_2\}$.

We define labels on the edges of Δ as follows: $lab_\Delta(e) \equiv lab_\Gamma(e)$ for all $e \in E(\Gamma) = E(\Delta)$.

Thus Δ is a graph labelled with $X^{\pm}$. In this situation we say that Δ is obtained from Γ by the *identification of a relator.* See Figure 2

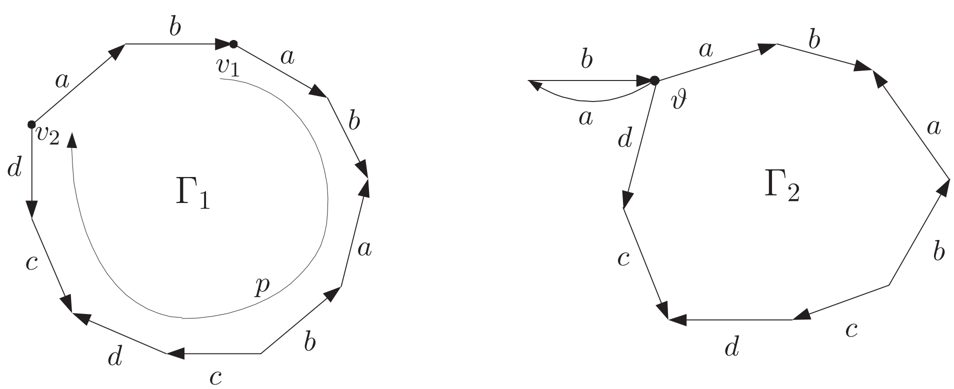

FIGURE 2. Let $G = gp\langle a, b, c, d \mid aba^{-1}b^{-1}cdc^{-1}d^{-1}\rangle$. Thus the graph Γ_2 is obtained from the graphs Γ_1 by the identification of the endpoints v_1 and v_2 of the path p labelled by the defining relator of G. Note that the resulting graph Γ_2 is not well-labelled (at ϑ).

Remark 3.9. There exists an epimorphism of pointed labelled graphs $\phi : (\Gamma, v_0) \to (\Delta, u_0)$ such that

$$\phi(v) = \begin{cases} v, & v \notin \{v_1, v_2\}; \\ \vartheta, & \text{otherwise.} \end{cases}$$

Thus $u_0 = \phi(v_0)$ and paths in Δ are images of paths in Γ. However, in order to simplify the notation we omit the use of the graph morphism

ϕ. We say that α' is a vertex/edge/path in Δ *corresponding* to the vertex/edge/path α in Γ, instead of saying that $\alpha' = \phi(\alpha)$ is the *image* of α in Δ. We treat Δ as a graph constructed from Γ in the combinatorial way described in Definition 3.8.

$\diamond$

Lemma 3.10. *Let $G = gp\langle X|R\rangle$ be a finitely presented group. Let Γ be a graph well-labelled with $X^{\pm}$. Let p be a freely reduced path in Γ with $lab(p) =_G 1$ such that $\iota(p) \neq \tau(p)$.*

Let Γ' be a graph obtained from Γ by the identification of the endpoints of p. Then $Lab(\Gamma, v_0) = Lab(\Gamma', v_0')$, where v_0 is the basepoint of Γ and v_0' is the corresponding basepoint of Γ'.

Proof. Let $q \in Loop(\Gamma, v_0)$. The identification of the endpoints of the path p keeps closed paths of Γ closed (because the graph morphism $\phi : (\Gamma, v_0) \to (\Gamma', v_0')$, see Remark 3.9, preserves endpoints). Thus the path q' in Γ' corresponding to the path q in Γ (that is obtained from q by the identification of the endpoints of p) is closed at v_0' if p is a subpath of q or if it is not a subpath of q. Thus $Loop(\Gamma, v_0) \subseteq Loop(\Gamma', v_0')$. Hence $Lab(\Gamma, v_0) \subseteq Lab(\Gamma', v_0')$.

Suppose now that $w \in Lab(\Gamma', v_0')$. Then there is $q' \in Loop(\Gamma', v_0')$ such that $lab(q') =_G w$. If q' exists in Γ (i.e. $q' \in Loop(\Gamma, v_0) \cap Loop(\Gamma', v_0'))$ then $w =_G lab(q') \in Lab(\Gamma, v_0)$.

Otherwise, $q' \in Loop(\Gamma', v_0') \setminus Loop(\Gamma, v_0)$. Let p' be the path corresponding to the path p in Γ' and $\vartheta \in V(\Gamma')$ be the vertex corresponding to the identified endpoints of the path p. Thus

$$\vartheta = \iota(p') = \tau(p'),\ lab(p) \equiv lab(p').$$

Hence the following is possible.

- p' is not a subpath of q'.

  Then there is a decomposition $q' = q_1' q_2' \dots q_k'$ such that

  $$\iota(q_1') = \tau(q_k') = v_0',\ \tau(q_i') = \iota(q_{i+1}') = \vartheta, 1 \leq i \leq k-1,$$

  where q_i' is a path in $\Gamma \cap \Gamma'$ and $q_i' q_{i+1}'$ is a path in Γ' which doesn't exist in Γ (see Figure 3). It means that $\tau(q_i')$ and $\iota(q_{i+1}')$ are different endpoints of the path p in Γ.

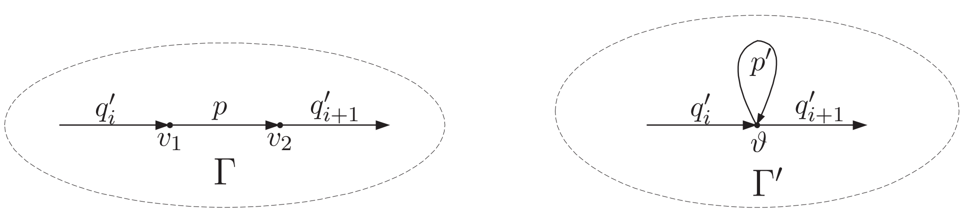

FIGURE 3.

Hence if $k = 1$ then the path q' is in $\Gamma \cap \Gamma'$. Therefore

$$w =_G lab(q') \in Lab(\Gamma, v_0).$$

Otherwise, let

$$p_i = \begin{cases} p, & \tau(q'_i) = \iota(p); \\ \overline{p}, & \tau(q'_i) = \tau(p). \end{cases}$$

Thus $q = q'_1 p_1 q'_2 p_2 \ldots p_{k-1} q'_k$ is a path in Γ closed at v_0. Since $lab(p_i) =_G 1$, we have

$$\begin{aligned} lab(q) &\equiv lab(q'_1) lab(p_1) lab(q'_2) lab(p_2) \ldots lab(p_{k-1}) lab(q'_k) \\ &=_G lab(q'_1) lab(q'_2) \ldots lab(q'_k) \equiv lab(q'). \end{aligned}$$

Thus $w =_G lab(q') =_G lab(q) \in Lab(\Gamma, v_0)$.

- p' is a subpath of q'.

  The occurrences of p' subdivide q' into a concatenation of paths of the form $q' = q'_1 p'_1 q'_2 p'_2 \ldots p'_{k-1} q'_k$, where $p'_i \in \{p', \overline{p'}\}$ and the paths q'_i don't involve p.

  For each $1 \leq i \leq k$, the path q'_i can be written as a decomposition of subpaths in $\Gamma \cap \Gamma'$, and the technique presented above (in the previous case) can be applied to it. Hence for all $1 \leq i \leq k$, there exists a path $q_i \subseteq \Gamma$ such that $\iota(q_i) = \iota(q'_i)$, $\tau(q_i) = \tau(q'_i)$ and $lab(q_i) =_G lab(q'_i)$.

  Let

$$p_i = \begin{cases} p, & p'_i = p'; \\ \overline{p}, & p'_i = \overline{p'}. \end{cases}$$

  Then $q = q_1 p_1 q_2 p_2 \ldots p_{k-1} q_k$ is a path in Γ closed at v_0. Moreover,

$$\begin{aligned} lab(q) &\equiv lab(q_1) lab(p_1) lab(q_2) lab(p_2) \ldots lab(p_{k-1}) lab(q_k) \\ &=_G lab(q'_1) lab(p'_1) lab(q'_2) lab(p'_2) \ldots lab(p'_{k-1}) lab(q'_k) \equiv lab(q'). \end{aligned}$$

Therefore $Lab(\Gamma, v_0) = Lab(\Gamma', v'_0)$.

$\diamond$

4. Subgroups and Covers

Below we recall the precise definitions of Cayley graphs and relative Cayley graphs based on [23], and present Lemma 4.1 from [11], which gives a characterization of their subgraphs.

The *Cayley graph* of the group presentation $G = gp\langle X | R \rangle$ is the oriented graph whose set of vertices is G and whose set of edges is $G \times X^{\pm}$, such that the edge (g, x) begins at the vertex g and ends at the vertex gx. We denote it $\boxed{Cayley(G)}$ omitting the specification of the group presentation of G, because along this paper it is fixed (see Section 5).

$Cayley(G)$ is a graph well-labelled with (the alphabet) $X^{\pm}$(that is a finite inverse automaton). Indeed, for each edge $(g, x) \in E(Cayley(G))$, $lab(g, x) = x$. Thus using the automata theoretic notation, we get $g \cdot x = gx$. For each path

$$p = (g, x_1)(gx_1, x_2) \cdots (gx_1 x_2 \cdots x_{n-1}, x_n)$$

in $Cayley(G)$, we obtain $lab(p) \equiv x_1x_2\cdots x_n \in (X^{\pm})^*$. That is $g\cdot(x_1x_2\cdots x_n) = gx_1x_2\cdots x_n$.

Let S be a subgroup of $G = gp\langle X|R\rangle$, and let G/S denote the set of right cosets of S in G. The *relative Cayley graph* of G with respect to S and the fixed group presentation $G = gp\langle X|R\rangle$, $\boxed{Cayley(G,S)}$ [3] , is an oriented graph whose vertices are the right cosets $G/S = \{Sg \mid g \in G\}$, the set of edges is $(G/S)\times X^{\pm}$, such that an edge (Sg,x) begins at the vertex Sg and ends at the vertex Sgx.

Therefore $Cayley(G,S)$ is a graph well-labelled with $X^{\pm}$ such that for each edge (Sg,x) in $Cayley(G,S)$, $lab(Sg,x) = x$. Using the automata theoretic notation, we get $(Sg)\cdot x = Sgx$. Thus for each path

$$p = (Sg,x_1)(Sgx_1,x_2)\cdots(Sgx_1x_2\cdots x_{n-1},x_n)$$

in $Cayley(G,S)$, $lab(p) \equiv x_1x_2\cdots x_n \in (X^{\pm})^*$ and $(Sg)\cdot(x_1\cdots x_n) = Sgx_1\cdots x_n$.

Any path p in $Cayley(G,S)$ which begins at $S\cdot 1$ [4] must end at $S\, lab(p)$, so p is a closed path at $S\cdot 1$ if and only if $lab(p)\in S$. Therefore,

$$Lab(Cayley(G,S), S\cdot 1) = S.$$

S acts on the Cayley graph of G by left multiplication, and $Cayley(G,S)$ can be defined as the quotient of the Cayley graph of G by this action.

Let K be the standard 2-complex presenting the group $G = \langle X|R\rangle$ (see [36], p. 157, for the precise definition). Thus K has one vertex, $|X|$ oriented edges and $|R|$ 2-cells. As is well known (see [36], pp.162-163), a geometric realization of a relative Cayley graph of G is a 1-skeleton of a topological cover of K. This enables us to call relative Cayley graphs of G, *"covers of G"*.

One sees, that $Cayley(G,S)$ is (the 1-skeleton of) a finite-sheeted cover (of K) if and only if it has a finite number of vertices, which means that S has finite index in G ([36], p. 162). However, the generating set X of G might be infinite, and then a finite-sheeted cover of G is an infinite graph. Thus the term "finite cover" is problematic in general. Nevertheless all groups which appear in this paper are finitely generated. This make it possible to use the above terminology without confusion.

The following result of Gitik [11] gives a characterization of subgraphs of relative Cayley graphs. In order to state it, the definitions below are needed.

A labelled graph Γ is G-*based*, if any path p in Γ with $lab(p) =_G 1_G$ is closed. Thus any G-based graph is necessarily well-labelled.

Let $x \in X^{\pm}$ and $v \in V(\Gamma)$. The graph Γ is x-*saturated* at v, if there exists $e \in E(\Gamma)$ with $\iota(e) = v$ and $lab(e) = x$. Γ is $X^{\pm}$-*saturated* if it is x-saturated for each $x \in X^{\pm}$ at each $v \in V(\Gamma)$.

[3] Whenever the notation $Cayley(G,S)$ is used, it always means that S is a subgroup of the group G and the presentation of G is fixed and clear from the context.

[4] We write $S\cdot 1$ instead of the usual $S1 = S$ to distinguish this vertex of $Cayley(G,S)$ as the basepoint of the graph.

Lemma 4.1 (Lemma 1.5 in [11]). *Let $G = gp\langle X|R\rangle$ be a group and let (Γ, v_0) be a graph well-labelled with $X^{\pm}$. Denote $Lab(\Gamma, v_0) = S$. Then*

- *Γ is G-based if and only if it can be embedded in $(Cayley(G, S), S\cdot 1)$,*
- *Γ is G-based and $X^{\pm}$-saturated if and only if it is isomorphic to $(Cayley(G, S), S\cdot\ 1)$.*

5. Normal Forms and Normal Core

Normal words in amalgams and normal paths in the corresponding labelled graphs are our basic tools. Below we recall their definitions. We define the new notion of the *normal core* of $Cayley(G, H)$. This graph is canonically associated with the subgroup H and will be constructed algorithmically in Section 8.

We start by fixing the notation. From now on whenever we refer to the group G we mean the amalgam $G = G_1 *_A G_2$, and whenever we refer to the group presentation of G we mean the following. We assume that the (free) factors are given by the finite group presentations

$$(1.a)\qquad G_1 = gp\langle X_1|R_1\rangle,\quad G_2 = gp\langle X_2|R_2\rangle \ \text{ such that }\ X_1^{\pm}\cap X_2^{\pm} = \emptyset.$$

A is a group such that there exist two monomorphisms

$$(1.b)\qquad \phi_1 : A \to G_1 \text{ and } \phi_2 : A \to G_2.$$

Thus G has a finite group presentation

$$(1.c)\qquad G = gp\langle X_1, X_2|R_1, R_2, \phi_1(A) = \phi_2(A)\rangle.$$

We put $X = X_1 \cup X_2$, $R = R_1 \cup R_2 \cup \{\phi_1(A) = \phi_2(A)\}$. Thus $G = gp\langle X|R\rangle$.

As is well known [23, 24, 33], the free factors embed in G. It enables us to identify A with its monomorphic image in each one of the free factors. Sometimes in order to make the context clear we'll use $\boxed{G_i \cap A}$, $i \in \{1, 2\}$, to denote the monomorphic image of A in G_i.

Normal Forms.

Definition 5.1 ([10, 23, 33]). *Let $G = G_1 *_A G_2$. We say that a word $g_1 g_2 \cdots g_n \in G$ is <u>in normal form</u> if:*

1. *$g_i \neq_G 1$ lies in one of the free factor of G,*
2. *g_i and g_{i+1} are in different factors of G,*
3. *if $n \neq 1$, then $g_i \notin A$.*

We call the sequence $(g_1, g_2, \ldots, g_n)$ a <u>normal decomposition</u> of the element $g \in G$, where $g =_G g_1 g_2 \cdots g_n$.

Any $g \in G$ has a representative in a normal form, [23, 24, 33]. If $g \equiv g_1 g_2 \cdots g_n$ is in normal form and $n > 1$, then the Normal Form Theorem [23] implies that $g \neq_G 1$.

By Serre [33], if g and h are two different words in normal form with normal decompositions $(g_1, g_2, \ldots, g_{n_1})$ and $(h_1, h_2, \ldots, h_{n_2})$, respectively,

then $g =_G h$ if and only if $n_1 = n_2 = n$ and there exist $a_i \in A,\ 1 \leq i \leq (n-1)$, such that

$$h_1 =_G g_1 a_1^{-1},\ h_j =_G a_{j-1} g_j a_j^{-1},\ 2 \leq j \leq n-1,\ h_n =_G a_{n-1} g_n.$$

The number n is unique for a given element g of G and it is called the *syllable length* of g (the subwords g_i are called the *syllables* of g). We denote it by $\boxed{length(g)}$. Notice that the number of letters in the word g is called the *length* of g and denoted $|g|$.

Let p be a path in the graph Γ, and let

$$p_1 p_2 \cdots p_n$$

be its decomposition into maximal monochromatic subpaths (i.e., subpaths labelled with either $X_1^{\pm}$ or $X_2^{\pm}$). Following the notation of Gitik, [11], we say that p is in *normal form* (by abuse of language, p is a *normal path*) if the word

$$lab(p) \equiv lab(p_1) lab(p_2) \cdots lab(p_n)$$

is in normal form.

If each p_i, $1 \leq i \leq n$ is a *geodesic* in $Cayley(G_j)$ (a *geodesic* is the shortest path joining two vertices) $j \in \{1, 2\}$, we say that p is in *strong normal form* (i.e. a *strong normal path*).

Normal Core.

Definition 5.2. *A vertex of $Cayley(G, H)$ is called <u>essential</u> if there exists a normal path closed at $H \cdot 1$ that goes through it.*

The <u>normal core</u> $(\Delta, H \cdot 1)$ of $Cayley(G, H)$ is the restriction of $Cayley(G, H)$ to the set of all <u>essential vertices</u>.

Remark 5.3. Note that the normal core $(\Delta, H \cdot 1)$ can be viewed as the union of all normal paths closed at $H \cdot 1$ in $(Cayley(G, H), H \cdot 1)$. Thus $(\Delta, H \cdot 1)$ is a connected graph with basepoint $H \cdot 1$.

Moreover, $V(\Delta) = \{H \cdot 1\}$ and $E(\Delta) = \emptyset$ if and only if H is the trivial subgroup. Indeed, H is not trivial iff there exists $1 \neq g \in H$ in normal form iff there exists $1 \neq g \in H$ such that g labels a normal path in $Cayley(G, H)$ closed at $H \cdot 1$, iff $E(\Delta) \neq \emptyset$.

$\diamond$

6. Reduced Precovers

The notion of *precovers* was defined by Gitik in [11] for subgroup graphs of amalgams. Such graphs can be viewed as a part of the corresponding covers of G, that explains the use of the term "precovers". Precovers are interesting from our point of view, because, by Lemma 6.9, they allow reading off normal forms on the graph. However these graphs could have (*redundant*) monochromatic components such that no closed normal path starting at the basepoint goes through them. Therefore, when looking for normal forms, our

attention can be restricted to precovers with no redundant monochromatic components – *reduced precovers*.

Precovers. We say that a vertex $v \in V(\Gamma)$ is *bichromatic* if there exist edges e_1 and e_2 in Γ with

$$\iota(e_1) = \iota(e_2) = v \text{ and } lab(e_i) \in X_i^{\pm},\ i \in \{1,2\}.$$

The set of bichromatic vertices of Γ is denoted by $VB(\Gamma)$. The vertex $v \in V(\Gamma)$ is called X_i-*mononochromatic* if all the edges of Γ beginning at v are labelled with $X_i^{\pm}$. We denote the set of X_i-monochromatic vertices of Γ by $VM_i(\Gamma)$ and put $VM(\Gamma) = VM_1(\Gamma) \cup VM_2(\Gamma)$.

A subgraph of Γ is called *monochromatic* if it is labelled only with $X_1^{\pm}$ or only with $X_2^{\pm}$. An X_i-*monochromatic component* of Γ ($i \in \{1,2\}$) is a maximal connected subgraph of Γ labelled with $X_i^{\pm}$, which contains at least one edge. Recall from Section 4, that by a *cover* of a group G we mean a relative Cayley graph of G corresponding to a subgroup of G.

Definition 6.1 ([11]). *Let $G = G_1 *_A G_2$. We say that a G-based graph Γ is a* <u>*precover*</u> *of G if each X_i-monochromatic component of Γ is a cover of G_i ($i \in \{1,2\}$).*

Remark 6.2. Note that by the above definition, a precover need not be a connected graph. However along this paper we restrict our attention only to connected precovers. Thus any time this term is used, we always mean that the corresponding graph is connected.

We follow the convention that a graph Γ with $V(\Gamma) = \{v\}$ and $E(\Gamma) = \emptyset$ determining the trivial subgroup (that is $Lab(\Gamma, v) = \{1\}$) is a (an empty) precover of G. $\diamond$

Example 6.3. Let $G = gp\langle x, y | x^4, y^6, x^2 = y^3 \rangle = \mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_6$.

Recall that G is isomorphic to $SL(2,\mathbb{Z})$ under the homomorphism

$$x \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ y \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}.$$

The graphs Γ_1 and Γ_3 on Figure 4 are examples of precovers of G with one monochromatic component and two monochromatic components, respectively.

Though the $\{x\}$-monochromatic component of the graph Γ_2 is a cover of $\mathbb{Z}_4$ and the $\{y\}$-monochromatic component is a cover of $\mathbb{Z}_6$, Γ_2 is not a precover of G, because it is not a G-based graph: $v \cdot (x^2 y^{-3}) = u$, while $x^2 y^{-3} =_G 1$.

The graph Γ_4 is not a precover of G because its $\{x\}$-monochromatic components are not covers of $\mathbb{Z}_4$. $\diamond$

Remark 6.4. Let Γ be a precover of G with $Lab(\Gamma, v_0) = H \leq G$. By Lemma 4.1, Γ is a subgraph of $Cayley(G, H)$. $\diamond$

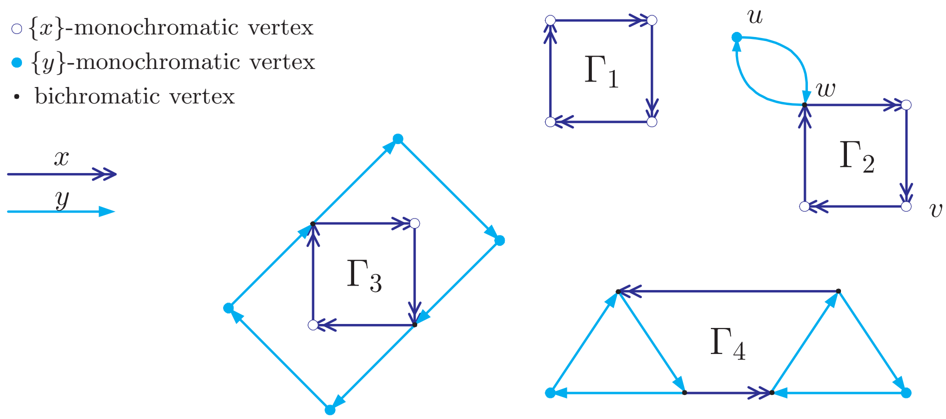

FIGURE 4.

Remark 6.5. Let $\phi : \Gamma \to \Delta$ be a morphism of labelled graphs. If Γ is a precover of G, then $\phi(\Gamma)$ is a precover of G as well.

Indeed, a morphism of labelled graphs preserves labels and commutes with endpoints. Thus $v \in V(\Gamma)$ is $X_1^{\pm}$-saturated/$X_2^{\pm}$-saturated/$X_1^{\pm} \cup X_2^{\pm}$-saturated implies $\phi(v) \in V(\Delta)$ is $X_1^{\pm}$-saturated/$X_2^{\pm}$-saturated/$X_1^{\pm} \cup X_2^{\pm}$-saturated. Furthermore, let $\phi(p)$ be a path in $\phi(\Gamma)$ with $lab(\phi(p)) =_G 1$. Therefore the path p in Γ satisfies $lab(p) = lab(\phi(p)) =_G 1$. Since Γ is a precover, p is closed. Hence the path $\phi(p)$ is closed in Δ. Therefore $\phi(\Gamma)$ is G-based. In particular, $\phi(\Gamma)$ is G_i-based, $i \in \{1, 2\}$. By Lemma 4.1, each X_i-monochromatic component of $\phi(\Gamma)$ is a cover of G_i. Hence $\phi(\Gamma)$ is a precover of G. $\diamond$

The graph Γ is called *compatible at a bichromatic vertex* v if for any monochromatic path p in Γ such that $\iota(p) = v$ and $lab(p) \in A$ there exists a monochromatic path t of a different color in Γ such that $\iota(t) = v$, $\tau(t) = \tau(p)$ and $lab(t) =_G lab(p)$. We say that Γ is *compatible* if it is compatible at all bichromatic vertices.

Example 6.6. The graphs Γ_1 and Γ_3 on Figure 4 are compatible. The graph Γ_2 does not possess this property because $w \cdot x^2 = v$, while $w \cdot y^3 = u$. Γ_4 is not compatible as well. $\diamond$

Remark 6.7 (Remark 2.11 in [11])**.** Precovers are compatible. $\diamond$

Remark 6.8 (Corollary2.13 in [11])**.** *Let Γ be a compatible graph. If all X_i-components of Γ are G_i-based, $i \in \{1, 2\}$, then Γ is G-based. In particular, if each X_i-component of Γ is a cover of G_i, $i \in \{1, 2\}$, and Γ is compatible, then Γ is a precover of G.*

Recall that our objective is to be able to read normal words on the constructed graph. The following lemma of Gitik shows that precovers are suitable for this purpose.

Lemma 6.9 (Lemma2.12 in [11])**.** *If Γ is a compatible graph, then for any path p in Γ there exists a path t in normal form which has the same endpoints and the same label (in G) as p.*

The statement of this lemma can be even extended when the graph Γ is a precover.

Lemma 6.10. *Let Γ be a precover of G. Let p be a path in Γ with $\iota(p) = v_1$, $\tau(p) = v_2$ and $lab(p) \equiv w$.*

Then for each normal word w' of syllable length greater than 1 such that $w' =_G w$ there exist a normal path p' in Γ with $\iota(p') = v_1$, $\tau(p') = v_2$ and $lab(p') \equiv w'$.

Proof. By Lemma 6.9, we can assume that the path p and the word w are normal. Let $p = p_1 \cdots p_k$ be a decomposition of p into maximal monochromatic paths ($k > 1$). Let C_i be the monochromatic component of Γ containing the subpath p_i of p ($1 \leq i \leq k$), that is $p_i \subseteq C_i \cap p$.

Let $lab(p_i) \equiv w_i$ ($1 \leq i \leq k$). Hence $w \equiv w_1 \cdots w_k$, where $(w_1, \ldots, w_k)$ is a normal (Serre) decomposition of w and $w_i \in G_{l_i}$ ($l_i \in \{1, 2\}$).

Let $w' \equiv w'_1 \cdots w'_m$ be a normal word with the normal (Serre) decomposition $(w'_1, \ldots, w'_m)$ such that $w =_G w'$. Therefore, by [33] p.4, $m = k$ and

$$w'_1 =_G w_1 a_{1_1}^{-1}, \;\; w'_i =_G a_{(i-1)_i} w_i a_{i_i}^{-1} \; (2 \leq i \leq k-1), \;\; w'_k =_G a_{(k-1)_k} w_k,$$

where $a_{1_j}, a_{i_j}, a_{(k-1)_j} \in A \cap G_{l_j}$ ($1 \leq j \leq k$) such that $a_{i_i} =_G a_{i_{(i+1)}}$.

Let $u_i = \tau(p_i)$ ($1 \leq i \leq k$). Thus $u_i \in VB(C_i) \cap VB(C_{i+1})$. Since Γ is a precover of G, C_i and C_{i+1} are covers of G_{l_i} and of $G_{l_{i+1}}$, respectively. That is they are $X_{l_i}^{\pm}$-saturated and $X_{l_{i+1}}^{\pm}$-saturated, respectively. Hence there are paths t_i in C_i and s_{i+1} in C_{i+1} starting at u_i and labelled by $a_{i_i}^{-1}$ and $a_{i_{i+1}}^{-1}$, respectively (see Figure 5). Since Γ is compatible (as a precover of G), $\tau(t_i) = \tau(s_{i+1})$. Hence there exists a path γ in Γ such that $\gamma = \gamma_1 \cdots \gamma_k$, where

$$\gamma_1 = p_1 t_1 \subseteq C_1, \;\; \gamma_i = \overline{s_i} p_i t_i \subseteq C_i \; (2 \leq i \leq k-1), \;\; \gamma_k = \overline{s_k} p_k \subseteq C_k.$$

Thus $\iota(\gamma) = v_1$, $\tau(\gamma) = v_2$ and

$$lab(\gamma_1) \equiv w_1 a_{1_1}^{-1}, \;\; lab(\gamma_i) \equiv a_{(i-1)_i} w_i a_{i_i}^{-1} \; (2 \leq i \leq k-1), \;\; lab(\gamma_k) \equiv a_{(k-1)_k} w_k.$$

Since $w'_i =_G lab(\gamma_i)$ ($1 \leq i \leq k$) and because the component C_i is $X_{l_i}^{\pm}$-saturated, there exists a path p'_i in C_i such that $\iota(p'_i) = \iota(\gamma_i)$ and $lab(p'_i) \equiv w'_i$. Moreover, $\tau(p'_i) = \tau(\gamma_i)$, because the component C_i is G_{l_i}-based. Therefore there exists a path $p' = p'_1 \cdots p'_k$ in Γ such that $\iota(p') = v_1$, $\tau(p') = v_2$ and $lab(p') \equiv w'$.

$\diamond$

Remark 6.11. When $length(w) = 1$ the statement of Lemma 6.10 need not be true. Thus, for example, the graph Γ, illustrated on Figure 6, is a precover of $G = gp\langle x, y | x^4, y^6, x^2 = y^3 \rangle = \mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_6$. There is a path p in

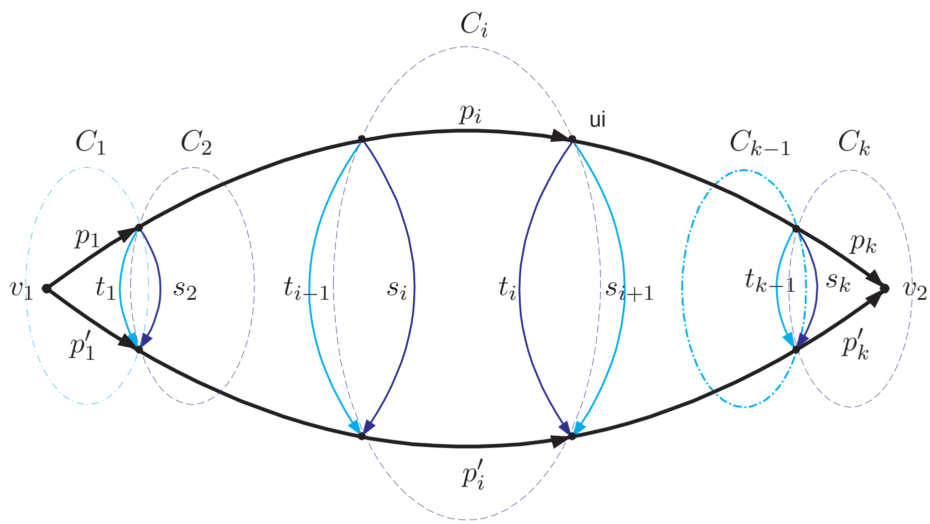

FIGURE 5.

Γ with $lab(p) \equiv x^2$ and $\iota(p) = \tau(p) = v_0$. However there is no path p' in Γ with the same endpoints as p and $lab(p') \equiv y^3$. $\diamond$

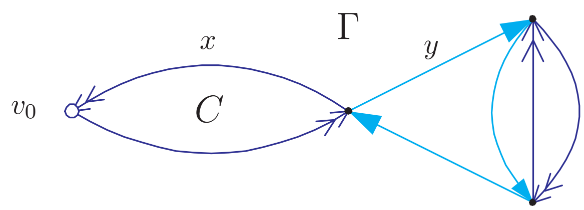

FIGURE 6. We use the same labelling as on Figure 4

Corollary 6.12. *Let p and p' be as in Lemma 6.10.*

If $G = G_1 *_A G_2$ is an amalgam of finite groups G_1 and G_2 then $p \subset N_d(p')$ and $p' \subset N_d(p)$, where $d = max(diameter(G_1), diameter(G_2))$.

Proof. Recall that a *group diameter* is the length of the longest geodesic in its Cayley graph.

Thus $d_j = diameter(G_j) = diameter(Cayley(G_j))$ $(j = 1, 2)$ is finite. Since each X_j-monochromatic component C of Γ is isomorphic to either $Cayley(G_j)$, $diameter(C) = d_j$. Let $d = max(d_1, d_2)$.

By the proof of Lemma 6.10, $p_i \subseteq C_i$ and $p'_i \subseteq C_i$. Thus $p_i \subset N_d(p'_i)$ and $p'_i \subset N_d(p_i)$ $(1 \leq i \leq k)$. Hence $p \subset N_d(p')$ and $p' \subset N_d(p)$. $\diamond$

However some monochromatic components of precovers may carry no "essential information" concerning normal forms. More precisely, if in a monochromatic component C of the precover Γ every path between any two bichromatic vertices of C is labelled by an element of A, then, evidently, no

normal path in Γ goes through this component, see Figure 7 and Example 6.14.

Below we are looking for an explicit characterization of such (*redundant*) monochromatic components of precovers. This enables us to present the new notion of a *reduced precover*, which is, roughly speaking, a precover with no redundant monochromatic components.

Redundant Monochromatic Components. Let Γ be a precover of G. Let C be a X_i-monochromatic component of Γ ($i \in \{1, 2\}$). Then A *acts* on $V(C)$ by right multiplication.

Let $v \in V(C)$, then the A-*orbit* of v is

$$A(v) = \{v \cdot a \mid a \in A\}.$$

Since Γ is a precover of G, it is compatible with A. Thus $v \in VB(C)$ if and only if $A(v) \subseteq VB(C)$. Hence bichromatic vertices of C are determined by the A-*action*. Moreover, $A(v) = VB(C)$ if and only if the bichromatic vertices of C form the unique A-orbit.

Claim 1. *For all $v_1, v_2 \in VB(C)$, $v_1 \cdot a = v_2$ implies $a \in A$ if and only if $VB(C) = A(\vartheta)$ and $Lab(C, \vartheta) = K \leq A$, for all $\vartheta \in VB(C)$.*

In other words, each path p in C ($C \subseteq \Gamma$) with $\iota(p), \tau(p) \in VB(C)$ satisfies $lab(p) \in A$ if and only if there exists a unique A-orbit of bichromatic vertices in C and $Lab(C, \vartheta) \leq A$, for all $\vartheta \in VB(C)$.

Proof of Claim 1. Assume first that $VB(C) = A(\vartheta)$ and $K = Lab(C, \vartheta) \leq A$. Let $v_1, v_2 \in VB(C)$. Since (C, ϑ) is isomorphic to $Cayley(G_i, K, K \cdot 1)$ and C has the unique A-orbit of bichromatic vertices, there exist $a_1, a_2 \in A$ such that $v_1 = (K \cdot 1) \cdot a_1 = Ka_1$ and $v_2 = (K \cdot 1) \cdot a_2 = Ka_2$. Thus

$$v_1 \cdot a = v_2 \;\Leftrightarrow\; (Ka_1) \cdot a = Ka_2 \;\Leftrightarrow\; a_1 a a_2^{-1} \in K.$$

Since $K \leq A$, we have $a \in A$.

Conversely, assume that for each pair of vertices $v_1, v_2 \in VB(C)$ each path p in C with $\iota(p) = v_1$ and $\tau(p) = v_2$ has $lab(p) \equiv a \in A$. In particular, if $v_1 = v_2 = \vartheta \in VB(C)$ then $\vartheta \cdot x = \vartheta$ implies $x \in A$. However $x \in Lab(C, \vartheta) = K$. Therefore $Lab(C, \vartheta) = K \leq A$. The equality $VB(C) = A(\vartheta)$ holds by the definition of A-orbits, because $\vartheta \in VB(C)$.

$\diamond$

Now we are ready to give a precise definition of the new notion of *redundant monochromatic components*.

Definition 6.13. *Let (Γ, v_0) be a precover of G. Let C be a X_i-monochromatic component of Γ ($i \in \{1, 2\}$). C is <u>redundant</u> if one of the following holds.*

1. *C is the unique monochromatic component of Γ (that is $\Gamma = C$) and $Lab(C, v_0) = \{1\}$ (equivalently, by Lemma 4.1, C is isomorphic to $Cayley(G_i)$).*

(2) Γ *has at least two distinct monochromatic components and the following holds.*

Let $\vartheta \in VB(C)$. *Let* $K = Lab(C, \vartheta)$ *(equivalently, by Lemma 4.1,* $(C, \vartheta) = (Cayley(G_i, K), K \cdot 1)$*). Then*

(i) $K \leq A$,

(ii) $VB(C) = A(\vartheta)$,

(iii) *either* $v_0 \notin V(C)$ *or,* $v_0 \in VB(C)$ *and* $K = \{1\}$.

Example 6.14. Let $G = gp\langle x, y | x^4, y^6, x^2 = y^3 \rangle = \mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_6$.

The graphs on Figure 7 are examples of precovers of G. The $\{x\}$-monochromatic component C of the graph Γ_1 is redundant, because (C, u) is isomorphic to $Cayley(\mathbb{Z}_4)$, that is $Lab(C, u) = \{1\}$, while $|VB(C)| = 2 = [\mathbb{Z}_4 : \mathbb{Z}_2]$ and $v_0 \notin V(C)$.

The $\{x\}$-monochromatic component D of the graph Γ_2 is redundant, because $Lab(D, v_0) = \{1\}$, while $v_0 \in VM(\Gamma_2)$.

However the graphs Γ_3 and Γ_4 have no redundant components. $\diamond$

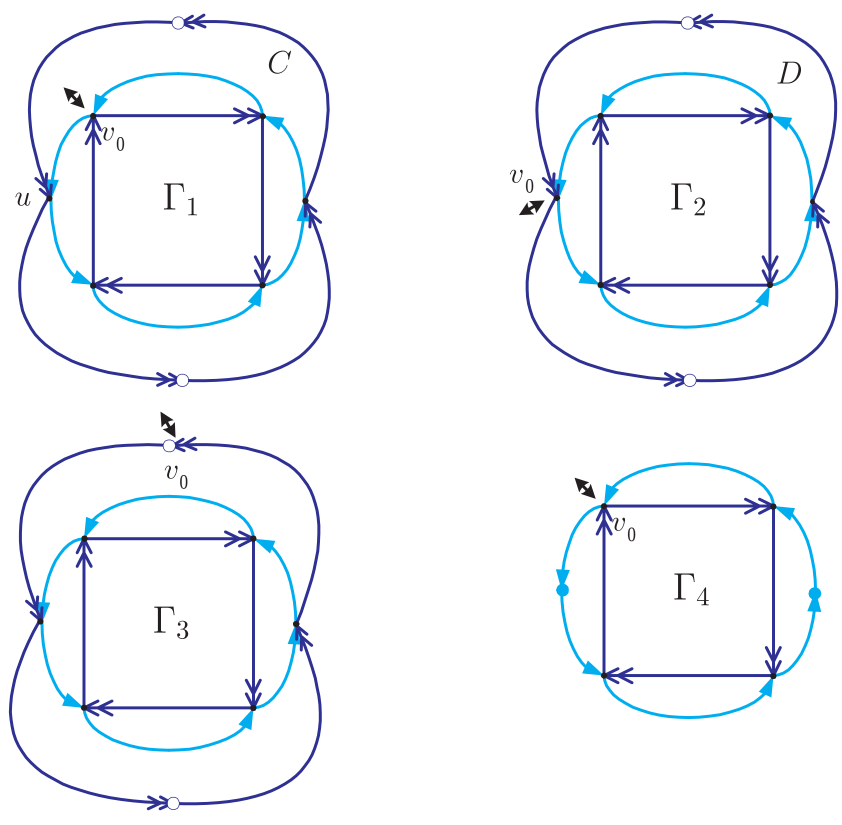

FIGURE 7. We use the same labelling as on Figure 4

Remark 6.15. Note that when the free factors G_1 and G_2 of the amalgam $G = G_1 *_A G_2$ are finite groups, then Claim 1 and Definition 6.13 can be restated in the following computational manner.

Recall that the A-*stabilizer* of v is

$$A_v = \{a \in A \mid v \cdot a = v\} \leq A.$$

As is well-known, the cosets of the stabilizer subgroup are in a one-to-one correspondence with the elements in the orbit

$$A(\vartheta) \sim A/A_\vartheta.$$

Thus

$$|A(\vartheta)| = [A : A_\vartheta].$$

Let $\vartheta \in VB(C)$. Let $K = Lab(C, \vartheta)$ (equivalently, $(C, \vartheta) = (Cayley(G_i, K), K \cdot 1)$). Hence $A_\vartheta = K \cap A$. Since $VB(C) = A(\vartheta)$ if and only if $|VB(C)| = |A(\vartheta)|$, the condition $K \leq A$ implies $VB(C) = A(\vartheta)$ if and only if $|VB(C)| = [A : K]$.

This enables us to replace the condition $VB(C) = A(\vartheta)$ in Claim 1 and in Definition 6.13 by its computational analogue $|VB(C)| = [A : K]$.

$\diamond$

Let us make the following assumption. From now on whenever we say that a path p in Γ *goes through the vertex* $v \in V(\Gamma)$, we mean that $v \in V(p)$. And whenever we say that a path p in Γ *goes through the monochromatic component* C in Γ, we mean that $E(p) \cap E(C) \neq \emptyset$. That is if $p = p_1 \cdots p_n$ is a decomposition of p into maximal monochromatic paths then there exists $1 \leq l \leq n$ such that C contains the subpath p_l ($p_l \subseteq p \cap C$ or, more precisely, $E(p_l) \subseteq E(p) \cap E(C)$).

Lemma 6.16. *Let* (Γ, v_0) *be a precover of* G*. Then* X_i*-monochromatic component* C *of* Γ $(i \in \{1, 2\})$ *is redundant if and only if no normal path* p *in* Γ *closed at* v_0 *goes through* C*.*

Proof. Let C be a X_i-monochromatic component of Γ ($i \in \{1, 2\}$). Let p be a path closed at v_0 that goes through C. Let $p = p_1 p_2 \cdots p_k$ be a decomposition of p into maximal monochromatic paths. Thus there exists $1 \leq j \leq k$ such that $p_j \subseteq C$.

If $k = 1$ then $p \subseteq C$ and $v_0 \in V(C)$. Thus p is normal if and only if $lab(p) \neq \{1\}$ if and only if $Lab(C, v_0) \neq \{1\}$ if and only if neither condition (1) nor condition (iii) in Definition 6.13 is satisfied.

Assume now that $k > 1$. The path p is normal if and only if $lab(p_j) \notin A$ for all $1 \leq j \leq k$. By Claim 1, this happens if and only if at least one of the conditions (i), (ii) in Definition 6.13 is not satisfied for the monochromatic component C.

Therefore p is normal if and only if C is not redundant.

$\diamond$

Now we show that removing of a redundant monochromatic component from a precover (Γ, v_0) leaves the resulting graph a precover and don't change the subgroup determined by the graph.

One can think of this procedure as an analogue of the "cutting hairs" procedure, presented by Stallings in [35], for subgroup graphs in the case of free groups. Indeed, a hair is cut from the graph because no freely reduced paths closed at the basepoint go through the hair. Similarly, when interested in normal paths closed at the basepoint of a precover, its redundant components can be erased, because no such paths go through them.

Let (Γ, v_0) be a precover of G. Let C be a redundant X_j-monochromatic component of Γ ($j \in \{1,2\}$). We say that the graph Γ' is obtained from the graph Γ by *removing of redundant X_j-monochromatic component C* , if Γ' is obtained by removing all edges and all X_j-monochromatic vertices of C, while keeping all its bichromatic vertices (see Figure 8). More precisely, if $\Gamma = C$ then we set $V(\Gamma') = \{v_0\}$, $E(\Gamma') = \emptyset$. Otherwise $V(\Gamma') = V(\Gamma) \setminus VM_j(C)$, where

$$\begin{aligned} VB(\Gamma') &= VB(\Gamma) \setminus VB(C), \\ VM_j(\Gamma') &= VM_j(\Gamma) \setminus VM_j(C), \ (1 \leq i \neq j \leq 2), \\ VM_i(\Gamma') &= VM_i(\Gamma) \cup VB(C). \end{aligned}$$

And

$$E(\Gamma') = E(\Gamma) \setminus E(C) \text{ and } lab_{\Gamma'}(e) \equiv lab_{\Gamma}(e) \ (\forall e \in E(\Gamma')).$$

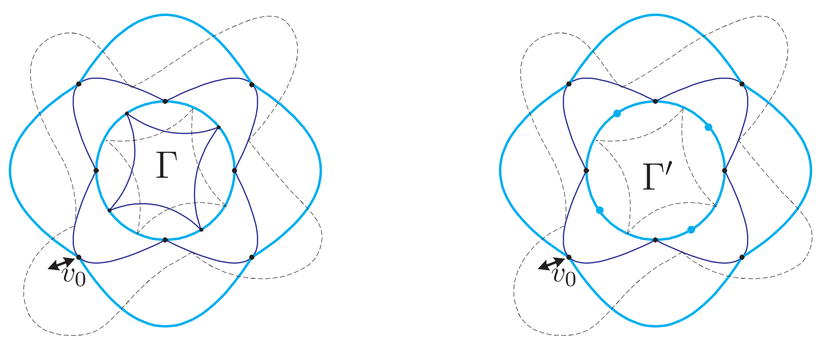

FIGURE 8. The closed grey curves represent G_1-monochromatic components. The closed black curves represent G_2-monochromatic components. The broken lines denote the rest of the graphs Γ and Γ', respectively. The small black dots are bichromatic vertices. The grey dots are X_1-monochromatic vertices.

Lemma 6.17. *Let (Γ, v_0) be a precover of G. Let Γ' be the graph obtained from the graph Γ by removing of a redundant X_j-monochromatic component C of Γ ($j \in \{1,2\}$).*

Then Γ' is a precover of G such that $Lab(\Gamma, v_0) = Lab(\Gamma', v_0')$, where v_0 is the basepoint of Γ and v_0' is the (corresponding) basepoint of Γ'.

Proof. If $\Gamma = C$ then Γ' is a precover, by the construction. Since C is redundant, $Lab(\Gamma, v_0) = \{1\}$. On the other hand, $Lab(\Gamma', v_0) = \{1\}$ as well. We are done.

Assume now that Γ has at least two monochromatic components.

Evidently, by the construction, Γ' is a precover of G. Indeed, since $VB(\Gamma') \subseteq VB(\Gamma)$ and Γ is compatible, then so is Γ'. Let $D \neq C$ be a monochromatic component of Γ. Then $D \subseteq \Gamma'$. Thus each X_i-monochromatic component of Γ' is a cover of G_i ($i \in \{1,2\}$). Hence Γ' is a precover of G, by Lemma 6.8. Note that $(\Gamma', v_0') \subseteq (\Gamma, v_0)$. Thus $Loop(\Gamma', v_0') \subseteq Loop(\Gamma, v_0)$ and we get $Lab(\Gamma', v_0') \subseteq Lab(\Gamma, v_0)$.

Let $w \in Lab(\Gamma, v_0)$. Then there is $t \in Loop(\Gamma, v_0)$ such that $lab(t) =_G w$. If no subpath of t is in C then t is also a path in Γ'. Therefore $lab(t) =_G w \in Lab(\Gamma', v_0')$.

Otherwise, there is a decomposition $t = t_1 q_1 t_2 q_2 \ldots q_{k-1} t_k$ such that $\iota(t_1) = \tau(t_k) = v_0$ and for all $1 \leq i \leq k$, q_i is a path in the component C and t_i is a path in Γ' with the normal decomposition $t_i = t_{i1} \cdots t_{im_i}$. Since $E(t_i) \cap E(C) = \emptyset$, the paths t_{im_i}, q_i and q_i, $t_{(i+1)1}$ are pairs of monochromatic paths of different colors. Thus the vertices $\tau(t_i) = \iota(q_i)$ and $\tau(q_i) = \iota(t_{i+1})$ are bichromatic vertices of Γ. Therefore, since C is redundant, Claim 1 implies that $lab(q_i) \in G_j \cap A$.

Let D be a X_l-monochromatic component of Γ such that t_{im_i} is a path in D, where $1 \leq j \neq l \leq 2$. Since Γ is a precover, D is a cover of G_l. Since the vertex $\iota(q_i)$ is bichromatic in Γ, while Γ is compatible and $lab(q_i) \in G_j \cap A$, there exists a path p_i in D such that

$$\iota(p_i) = \iota(q_i), \ \tau(p_i) = \tau(q_i) \text{ and } lab(p_i) =_G lab(q_i).$$

Thus the path $t' = t_1 p_1 t_2 p_2 \ldots p_{k-1} t_k$ is a closed path at v_0' in Γ' with $lab(t') =_G lab(t)$. Therefore $w \equiv lab(t) =_G lab(t') \in Lab(\Gamma', v_0')$.

Hence $Lab(\Gamma) = Lab(\Gamma')$.

Proceeding in the same manner as in the construction of t' (in Γ') from the path t (in Γ), one can shows that any two vertices of Γ remain connected by a path in Γ'. More precisely, given a pair of vertices v and w in Γ and given a path s in Γ connecting them, one can construct an appropriate path s' in Γ' such that $\iota(s') = \iota(s) = v$, $\tau(s') = \tau(s) = w$ and $lab(s') =_G lab(s)$. Therefore the graph Γ' is connected.

$\diamond$

Reduced Precovers.

Definition 6.18. *A precover* (Γ, v_0) *of* G *is called* <u>*reduced*</u> *if and only if the following holds*

1. (Γ, v_0) *has no redundant monochromatic components.*
2. *If there exists a* X_i*-monochromatic component* C *of* Γ *(*$i \in \{1,2\}$*) such that* $v_0 \in V(C)$ *and* $K \cap A \neq \{1\}$*, where* $K = Lab(C, v_0)$ *(equivalently,* $(C, v_0) = (Cayley(G_i, K), K \cdot 1)$*), then there exists*

a X_j-monochromatic component D of Γ $(1 \leq i \neq j \leq 2)$ such that $v_0 \in V(D)$ and $K \cap A =_G L \cap A$, where $L = Lab(D, v_0)$ *(equivalently, $(D, v_0) = (Cayley(G_i, L), L \cdot 1)$).*

Remark 6.19. Note that condition (2) in the above definition merely says that if $A \cap H \neq \{1\}$ then $v_0 \in VB(\Gamma)$, where $H = Lab(\Gamma, v_0)$.

Therefore if Γ has the unique X_i-monochromatic component C (that is $\Gamma = C$, $i \in \{1, 2\}$) then H is a nontrivial subgroup of G_i such that $A \cap H = \{1\}$.

If $V(\Gamma) = \{v_0\}$ and $E(\Gamma) = \emptyset$ then Γ is a reduced precover, by the above definition, with $Lab(\Gamma, v_0) = \{1\}$ ⋄

Example 6.20. Let $G = gp\langle x, y | x^4, y^6, x^2 = y^3 \rangle = \mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_6$.

The precovers Γ_1 and Γ_2 from Figure 7 are not reduced because they have redundant components C and D, respectively (see Example 6.14). The graphs Γ_3 and Γ_4 from the same figure are reduced precover of G because they are precovers with no redundant components and with a bichromatic basepoint.

The precover Γ on Figure 9 is not a reduced precover of G though it has no redundant components. The problem now is the $\{x\}$-monochromatic component C of Γ because $Lab(C, v_0) = \langle x^2 \rangle$, while the basepoint v_0 is a $\{x\}$-monochromatic vertex. It is easy to see that the graph Γ' obtained from Γ by gluing at v_0 the appropriate $\{y\}$-monochromatic component D with $Lab(D, v_0) = \langle y^3 \rangle$ is a reduced precover of G, by Definition 6.18. ⋄

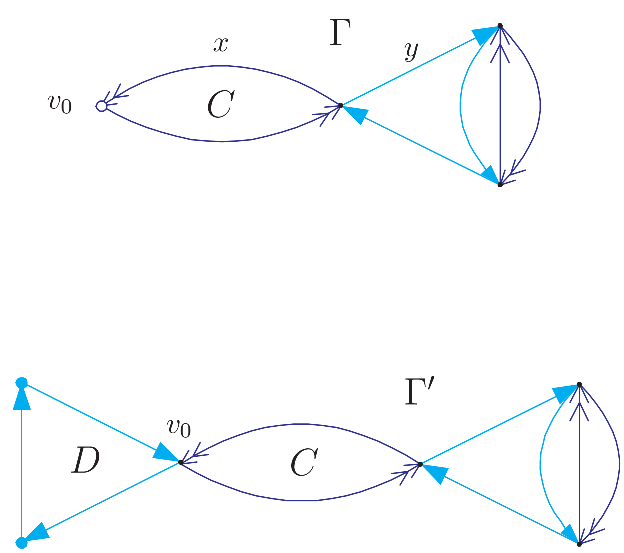

FIGURE 9. We use the same labelling as on Figure 4

Let (Γ, v_0) be a precover of G with no redundant components, which is not a reduced precover. Hence $v_0 \in VM_l(\Gamma)$ $(l \in \{1, 2\})$ and the assumption

of condition (2) in Definition 6.18 holds, that is Γ has a X_l-monochromatic component C with $Lab(C, v_0) = K$ such that $L = K \cap A$ is a nontrivial subgroup of A.

Thus (Γ, v_0) can be "reconstructed" in the obvious way (see Figure 9 and Example 6.20) such that the resulting graph is a reduced precover of G determining the same subgroup as the graph (Γ, v_0) does.

Let (Γ', v_0') be the graph obtained by taking a disjoint union of the graphs (Γ, v_0) and $(Cayley(G_j, L), L \cdot 1)$ $(1 \leq j \neq l \leq 2)$ via the identification of $L \cdot 1$ with v_0 and the identification of the X_j-monochromatic vertices La of $(Cayley(G_j, L), L \cdot 1)$, for all $a \in (G_j \cap A) \setminus L$, with the X_l-monochromatic vertices $v_0 \cdot b$ of C, where $b \in G_l \cap A$ such that $b =_G a$. The following lemma is a straightforward result of this construction.

Lemma 6.21. *(Γ', v_0') is a reduced precover of G with $Lab(\Gamma, v_0) = Lab(\Gamma', v_0')$, where v_0 is the basepoint of Γ and v_0' is the (corresponding) basepoint of Γ'.*

Proof. Obviously, by construction, Γ' is well-labelled, compatible with A and each monochromatic component of Γ' is a cover of either G_1 or G_2. Thus Γ' is a precover of G. Moreover, Γ' has no redundant components and condition (2) from Definition 6.18 is satisfied. Hence (Γ', v_0') is a reduced precover of G.

By construction, Γ and $Cayley(G_j, L, L \cdot 1)$ embed in Γ'. Hence $(\Gamma, v_0) \subseteq (\Gamma', v_0')$, thus $Loop(\Gamma, v_0) \subseteq Loop(\Gamma', v_0')$. Therefore $Lab(\Gamma, v_0) \subseteq Lab(\Gamma', v_0')$.

Let $u \in Lab(\Gamma', v_0')$. Hence there is $t' \in Loop(\Gamma', v_0')$ such that $lab(t') =_G u$. If t' is a path in Γ therefore

$$lab(t') =_G u \in Lab(\Gamma, v_0).$$

Otherwise there is a decomposition

$$t' = t_1' q_1 t_2' q_2 \dots q_{k-1} t_k'$$

such that $\iota(t_1') = \tau(t_k') = v_0'$, and for all $1 \leq i \leq k$, $t_i' \subseteq \Gamma$ and q_i is a path in Γ' which doesn't exist in Γ.

Thus for all $1 \leq i \leq k$, q_i is a path in $Cayley(G_j, L, L \cdot 1)$ such that $\iota(q_i) = v_{i_1}$ and $\tau(q_i) = v_{i_2}$ are the common images in Γ' of the vertices $w_{i_1}, w_{i_2} \in \{v_0 \cdot a \mid a \in A \setminus L\}$ of C and the vertices $u_{i_1}, u_{i_2} \in \{La \mid a \in A \setminus L\}$ of $Cayley(G_j, L, L \cdot 1)$, respectively. By abuse of notation, we write $v_0 \cdot a_{i_1} = w_{i_1} = \iota(q_i) = u_{i_1} = La_{i_1}$ and $v_0 \cdot a_{i_2} = w_{i_2} = \tau(q_i) = u_{i_2} = La_{i_2}$, where $a_{i_1},\ a_{i_2} \in A \setminus L$.

Since $(La_{i_1}) \cdot lab(q_i) = La_{i_2}$, there exists $b \in L$ such that $lab(q_i) =_G a_{i_1}^{-1} b a_{i_2}$. Hence $w_{i_1} \cdot (a_{i_1}^{-1} b a_{i_2}) = (v_0 \cdot b) \cdot a_{i_2} = v_0 \cdot a_{i_2} = w_{i_2}$, because $b \in L \leq K$. Therefore there exists a path q_i' in C (that is in Γ) such that

$$\iota(q_i') = w_{i_1},\ \tau(q_i') = w_{i_2},\ lab(q_i') =_G lab(q_i).$$

Thus there exists a path t in Γ such that $t = t'_1 q'_1 t'_2 q'_2 \ldots q'_k t'_k$. Therefore

$$\begin{aligned} lab(t) &\equiv lab(t'_1)lab(q'_1)lab(t'_2)lab(q'_2)\ldots lab(q'_k)lab(t'_k) \\ &=_G lab(t'_1)lab(q_1)lab(t'_2)lab(q_2)\ldots lab(q_k)lab(t'_k) \\ &\equiv lab(t'). \end{aligned}$$

Since $lab(t') \in Lab(\Gamma, v_0)$, we have $Lab(\Gamma) = Lab(\Gamma')$.

$\diamond$

Lemma 6.22. *Let (Γ, v_0) be a reduced precover of G. Then for each $v \in V(\Gamma)$ there exists a normal path p in Γ closed at v_0 such that $v \in V(p)$.*

Proof. Let C be a X_i-monochromatic component of Γ ($i \in \{1,2\}$) such that $v \in V(C)$.

Since C is not redundant, by Lemma 6.16, there exists a normal path q in Γ closed at v_0 that goes through C. Let $q = q_1 \cdots q_m$ be a normal decomposition of q into maximal monochromatic paths. Assume that $q_l \subseteq q \cap C$ ($1 \leq l \leq m$). Let $v_1 = \iota(q_l)$ and $v_2 = \tau(q_l)$.

If $v \in V(q_l)$ then $p = q$ is the desired path. Otherwise, we proceed in the following way. Assume first that $m = 1$. Then, by the proof of Lemma 6.16, $Lab(C, v_0) \neq \{1\}$. Let t be a path in C with $\iota(t) = v_0$, $\tau(t) = v$ and $lab(t) \equiv g$. Hence $Lab(C, v) = g^{-1}Lab(C, v_0)g \neq \{1\}$. Therefore there exists a nonempty path $q' \in Loop(C, v)$ such that $lab(q') \neq_G 1$. Therefore $tq'\overline{t} \in Loop(C, v_0)$ and $lab(tq'\overline{t}) \neq_G 1$. Thus $lab(p_v p \overline{p_v})$ is a normal word, because it is a nonempty word of syllable length 1, which is not the identity in G. Hence $p = tq'\overline{t}$ is the desired normal path in Γ closed at v_0 that goes through v.

Assume now that $m > 1$. Let t_j be paths in C with $\iota(t_j) = v_j$ to $\tau(t_j) = v$ ($j \in \{1,2\}$), see Figure 10. Let $t = t_1\overline{t_2}$. Since $deg_\Gamma(v) \geq 2$ (Γ is a precover of G), we can assume that t is freely reduced.

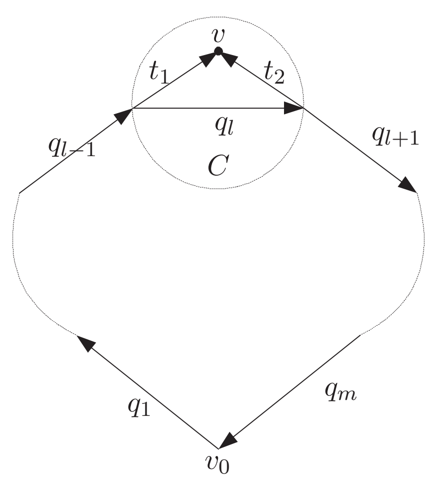

FIGURE 10.

If $lab(t) \notin A$ then the path $p = q_1 \cdots q_{l-1} t q_{l+1} \cdots q_m$ is the desired normal path in Γ closed at v_0 which goes through v.

If $lab(t) \in A$ then $lab(t\overline{q_l}t) \equiv lab(t)lab(\overline{q_l})lab(t) \notin A$, because $lab(q_l) \notin A$. Hence $p = q_1 \cdots q_{l-1}(t\overline{q_l}t)q_{l+1} \cdots q_m$ is the desired normal path in Γ closed at v_0 which goes through v.

$\diamond$

7. The Main Theorem

Let H be a finitely generated subgroup of the amalgam $G = G_1 *_A G_2$. As was discussed in the previous sections, there exist labelled graphs which can be 'naturally' associated with H. Thus the examples of such graphs are the normal core of $Cayley(G, H)$, on the one hand, and a reduced precover of G, (Γ, v_0), with $Lab(\Gamma, v_0) = H$, on the other.

Below we prove that normal cores and reduced precovers determining the same subgroup H are the same. That is they define the same part of $Cayley(G, H)$ in different ways: the normal core defines it theoretically, while the reduced precover characterizes it constructively.

Theorem 7.1 (The Main Theorem)**.** *Let H be a finitely generated subgroup of the amalgam $G = G_1 *_A G_2$. Then up to isomorphism there exists a unique reduced precover of G determining H, which is the normal core $(\Delta, H \cdot 1)$ of $Cayley(G, H)$.*

We separate the proof of the main theorem into two parts. First we prove that if there exists a reduced precover of G determining the subgroup H then up to isomorphism it is unique. This statement follows from Theorem 7.2. Then we prove (Theorem 7.5) that given a finitely generated subgroup H of G there exists a reduced precover determining H, which is precisely the normal core of $Cayley(G, H)$.

Let (Γ, v_0) be a pointed graph labelled with $X^{\pm}$. Define

$$\mu : (\Gamma, v_0) \rightarrow (Cayley(G, S), S \cdot 1)$$

such that

$$\forall v \in V(\Gamma),\ \mu(v) = (S \cdot 1) \cdot lab(p) = S(lab(p)),$$

where p is a path in Γ with $\iota(p) = v_0$, $\tau(p) = v$, and

$$\forall e \in E(\Gamma),\ \mu(e) = (\mu(\iota(e)), lab(e)).$$

In the proof of Lemma 4.1 (Lemma 1.5 in [11]) Gitik shows that μ is a morphism of labelled pointed graphs which is injective if Γ is G-based. Hence if Γ is a precover of G, then the morphism μ is an embedding. We are interested in an identification of the monomorphic image $\mu(\Gamma)$ inside $Cayley(G, S)$.

Theorem 7.2. *Let (Γ, v_0) be a reduced precover of G.*
Let $(\Delta, H \cdot 1)$ be the normal core of $Cayley(G, H)$, where $H = Lab(\Gamma, v_0)$. Then $\mu(\Gamma, H \cdot 1) = (\Delta, H \cdot 1)$.

Proof. If $V(\Gamma) = \{v_0\}$ and $E(\Gamma) = \emptyset$ then $H = \{1\}$ by Remark 3.4. Therefore, by Remark 5.3, $V(\Delta) = \{H \cdot 1\}$ and $E(\Gamma) = \emptyset$. We are done.

First we show that $\mu(\Gamma, v_0) \subseteq (\Delta, H \cdot 1)$. Let $u \in V(\mu(\Gamma)) = \mu(V(\Gamma))$. Hence $u = \mu(v)$, where $v \in V(\Gamma)$. Without loss of generality, we can assume that $v \neq v_0$, otherwise the statement is trivial ($\mu(v_0) = H \cdot 1 \in V(\Delta)$), because μ is a morphism of pointed graphs.

By Lemma 6.22, there exists a normal path p in Γ closed at v_0 such that $v \in V(p)$. Since graph morphisms commute with ι, τ and preserve labels, $\mu(p)$ is a normal path in $Cayley(G, H)$ that goes through the vertex $\mu(v) = u$. Thus $\mu(p)$ is a path in the normal core $(\Delta, H \cdot 1)$ and $\mu(v) = u \in V(\Delta)$. Therefore $V(\mu(\Gamma)) \subseteq V(\Delta)$. Since graph morphisms commute with ι, τ and preserve labels, we conclude that $\mu(\Gamma, v_0) \subseteq (\Delta, H \cdot 1)$.

Now we prove that $\mu(\Gamma, v_0) \supseteq (\Delta, H \cdot 1)$.

Let $\sigma \in V(\Delta)$. Then there is a normal path δ in Δ closed at $H \cdot 1$ in $Cayley(G, H)$ such that $\sigma \in V(\delta)$. Thus $lab(\delta) \in H$ is a word in normal form. Hence there exists a path p in Γ closed at v_0 with $lab(p) =_G lab(\delta)$. Since Γ is a precover, by Lemma 6.10, there exists a path p' in Γ closed at v_0 with $lab(p') \equiv lab(\delta)$. Therefore $\delta = \mu(p')$. Hence there exists $v \in V(p')$ such that $\sigma = \mu(v)$.

Therefore $V(\Delta) \subseteq V(\mu(\Gamma))$. Since graph morphisms commute with ι, τ and preserve labels, we conclude that $(\Delta, H \cdot 1) \subseteq \mu(\Gamma, v_0)$. Hence $(\Delta, H \cdot 1) = \mu(\Gamma, v_0)$.

$\diamond$

Corollary 7.3. *Following the notation of Theorem 7.2, μ is an isomorphism of (Γ, v_0) and $(\Delta, H \cdot 1)$.*

Corollary 7.4. *Any pair of reduced precovers of G determining the same subgroup are isomorphic.*

Theorem 7.5. *Let H be a finitely generated subgroup of G. Then the normal core $(\Delta, H \cdot 1)$ of $Cayley(G, H)$ is a reduced precover of G with $Lab(\Delta, H \cdot 1) = H$.*

Proof. Without loss of generality, we can assume that $H \neq \{1\}$, because otherwise, by Remark 5.3, the statement is trivial.

By definition, a well-labelled graph Γ is a precover of G if it is G-based and each X_i-monochromatic component of Γ ($i \in \{1, 2\}$) is a cover of G_i.

Since Δ is a subgraph of $Cayley(G, H)$, Δ is well-labelled with $X_1^{\pm} \cup X_2^{\pm}$ and G-based. Therefore each X_i-monochromatic component of Δ is G_i-based ($i \in \{1, 2\}$). By Lemma 4.1, in order to conclude that each such component is a cover of G_i, we have to show that it is $X_i^{\pm}$-saturated.

Let C be a X_i-monochromatic component of Δ ($i \in \{1, 2\}$). Let $v \in V(C)$ and $x \in X_i$. Let C' be the X_i-monochromatic component of $Cayley(G, H)$ such that $C \subseteq C'$. Therefore there is $e \in E(C')$ such that $lab(e) \equiv x$, $\iota(e) = v$ and $v_x = \tau(e) \in V(C')$.

Since $v \in V(C) \subseteq V(\Delta)$, there is a normal form path p in Δ that goes through v. If $e \in E(p)$ then we are done. Otherwise, let

$$p = p_1 \cdots p_{l-1} q p_{l+1} \cdots p_k$$

be a normal decomposition of p into maximal monochromatic subpaths, such that $p \cap C = q$, $v \in V(q)$ and $lab(q) \in G_i \setminus A$ (see Figure 11).

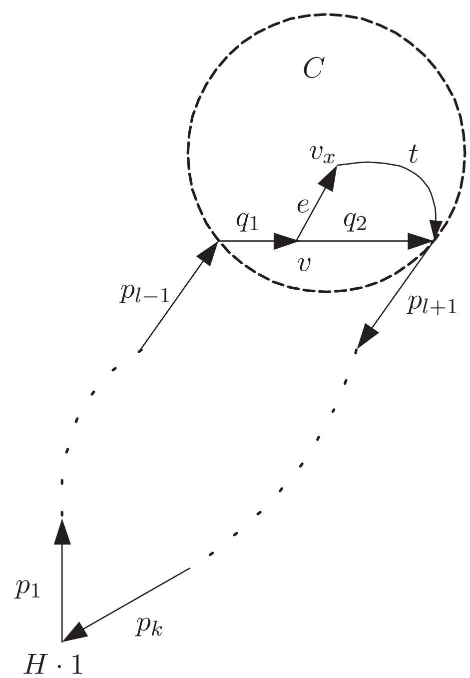

FIGURE 11. An auxiliary figure for the proof of Lemma 8.3

Let $\iota(q) = v_1$, $\tau(q) = v_2$, $q = q_1 q_2$ such that $\tau(q_1) = v = \iota(q_2)$. Let t be a path in C' with $\iota(t) = v_x$, $\tau(t) = v_2$. Thus $\iota(q_1 e t) = v_1 = \iota(q)$ and $\tau(q_1 e t) = v_2 = \tau(q)$. (Since the graph $Cayley(G, H)$ has no hairs, without loss of generality, we can assume that the path $q_1 e t$ is freely reduced.)

Let

$$q' = \begin{cases} q_1 e t, & lab(q_1 e t) \notin A; \\ (q_1 e t)\overline{q}(q_1 e t), & lab(q_1 e t) \in G_1 \cap A. \end{cases}$$

Thus the path q' has the same endpoints as q, $v_x \in V(q')$ and $lab(q') \in G_i \setminus A$. Hence $p = p_1 \cdots p_i q' p_{i+1} \cdots p_k$ is a normal form path that goes through the vertex v_x. Therefore $p \subseteq \Delta$ and $e \in E(C) \subseteq E(\Delta)$. As this is true for every $x \in X_i$, the vertex v is $X_i^{\pm}$-saturated. Hence, by Definition 6.1, Δ is a precover of G.

By Lemma 6.16, Δ has no redundant monochromatic components, because for each $v \in V(\Delta)$ there is a path in normal form closed at $H \cdot 1$ that goes through v.

Assume now that C is a X_i-monochromatic component of Δ ($i \in \{1, 2\}$) such that $(C, H \cdot 1)$ is isomorphic to $Cayley(G_i, K, K \cdot 1)$, where $K \cap A$ is

a nontrivial subgroup of A. Then there exists a nonempty normal path p in $C \subseteq \Delta$ closed at $H \cdot 1$ with $lab(p) \equiv w \in K \cap A \cap G_i$. Since $\{1\} \neq K \cap A \leq A$, there exists $1 \neq u \in G_j \cap A$ $(1 \leq i \neq j \leq 2)$ such that $w =_G u$. Thus the syllable length of the words w and u is equal to 1. Therefore these words are in normal form.

The graph $Cayley(G, H)$ is $X^{\pm}$ saturated and compatible with A. Thus $H \cdot 1 \in VB(Cayley(G, H))$ and therefore there exists a path q in $Cayley(G, H)$ closed at $H \cdot 1$ with $lab(q) \equiv u$. Hence $q \subseteq \Delta$, because u is in normal form. Since Δ is a precover of G, $D \subseteq \Delta$, where D is a X_j-monochromatic component of $Cayley(G, H)$ such that $q \subseteq D$ and $(D, H \cdot 1)$ is isomorphic to $Cayley(G_j, L, L \cdot 1)$.

Since, Δ is compatible with A (as a subgraph of $Cayley(G, H)$), $L \cap A =_G K \cap A$. Then, by Definition 6.18, $(\Delta, H \cdot 1)$ is a reduced precover of G.

$\diamond$

Proof of The Main Theorem. The statement is an immediate consequence of Corollary 7.4, Theorem 7.5 and Lemma 8.3.

$\diamond$

8. The Algorithm

Let H be a finitely generated subgroup of an amalgam $G = G_1 *_A G_2$. By Definition 5.2 and Remark 5.3, the normal core of $Cayley(G, H)$ depends on H itself and not on the set of subgroup generators, therefore this graph is canonically associated with the subgroup H. Hence it can be exploited to study certain properties of H.

In Lemma 8.3 we prove that when the factors G_1 and G_2 are finite groups, the normal core of $Cayley(G, H)$ is a finite graph, which is completely defined by H. Thus, evidentially, it can be constructed. Our main theorem (Theorem 7.1) hints the way. Indeed, by Theorem 7.1, the normal core of $Cayley(G, H)$ is the unique reduced precover of G determining H. Therefore in order to construct the normal core of $Cayley(G, H)$ we should take the 'right bunch' of copies of relative Cayley graphs of the free factors, glue them to each other according to the amalgamation, and verify that the obtained precover is reduced. If not then it can be converted to a reduced precover using Lemmas 6.17 and 6.21.

The precise algorithm, the proof of its finiteness and validity, and the complexity analysis are presented in the current section.

Our proof of the finiteness of the normal core is based on the following result of Gitik [10].

Definition 8.1 ([10]). *Let $G = gp\langle X | R \rangle$. Let*

$$\pi_S : Cayley(G) \to Cayley(G, S)$$

be the projection map such that $\pi_S(g) = Sg$ and $\pi_S(g, x) = (Sg, x)$.

A geodesic in $Cayley(G, S)$ is the image of a geodesic in $Cayley(G)$ under the projection π_S. The geodesic core of $Cayley(G, S)$, $\boxed{Core(G, S)}$, is the union of all closed geodesics in $Cayley(G, S)$ beginning at the vertex $S \cdot 1$.

Lemma 8.2 (Lemma 1.5 in [10])**.** *A subgroup S of a group G is K-quasiconvex in G if and only if $Core(G, S)$ belongs to the K-neighborhood of $S \cdot 1$ in $Cayley(G, S)$.*

Lemma 8.3. *Let H be a finitely generated subgroup of $G = G_1 *_A G_2$.*

If G_1 and G_2 are finite groups. Then the normal core $(\Delta, H \cdot 1)$ of $Cayley(G, H)$ is a finite graph.

Proof. Since the group G is locally-quasiconvex ([16]), the subgroup H is quasiconvex. Therefore, $Core(G, H)$ is a finite graph, by Lemma 8.2.

Let $\overline{\gamma}$ be a closed normal path starting at $H \cdot 1$ in $(\Delta, H \cdot 1) \subset (Cayley(G, H), H \cdot 1)$. Thus $\overline{\gamma}$ is the image under the projection map π_H (see Definition 8.1) of the normal path γ in $Cayley(G)$ whose endpoints and the label are in H. That is $lab(\gamma) \equiv h \in H$.

Since G_1 and G_2 are finite, they are quasiconvex subgroups of the hyperbolic group G. Thus the conditions of Lemma 1.1 are satisfied. Let $\epsilon \geq 0$ be the constant from Lemma 1.1. Let δ be a geodesic in $Cayley(G)$ with the same endpoints as γ. By Lemma 1.1, there exists a strong normal path δ' in $Cayley(G)$ with the same endpoints as δ such that $\delta' \subset N_\epsilon(\delta)$ and $\delta \subset N_\epsilon(\delta')$ (see Figure 12).

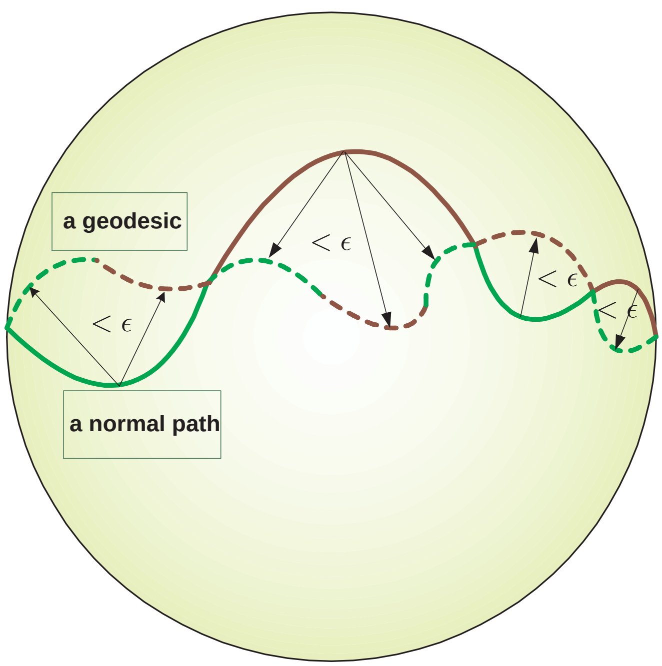

FIGURE 12.

Thus γ and δ' are two normal form paths in $Cayley(G)$ with the same endpoints. Therefore $lab(\gamma) =_G lab(\delta')$. By Corollary 6.12, $\gamma \subset N_d(\delta')$ and $\delta' \subset N_d(\gamma)$, where $d = max(diameter(G_1), diameter(G_2))$.

Let $\epsilon' = \epsilon + d$. Then $\gamma \subset N_{\epsilon'}(\delta)$ and $\delta \subset N_{\epsilon'}(\gamma)$. Since the projection map π_H does not increase distances, and it maps γ onto $\overline{\gamma}$ in $(\Delta, H \cdot 1) \subseteq (Cayley(G, H), H \cdot 1)$ and δ onto $\overline{\delta}$ in $Core(G, H) \subseteq (Cayley(G, H), H \cdot 1)$, we have $\overline{\gamma} \subset N_{\epsilon'}(\overline{\delta})$ and $\overline{\delta} \subset N_{\epsilon'}(\overline{\gamma})$.

This implies that $Core(G, H) \subset N_{\epsilon'}(\Delta)$ and $\Delta \subset N_{\epsilon'}(Core(G, H))$. Since $Core(G, H)$ is a finite graph we conclude that the graph $(\Delta, H \cdot 1)$ is finite as well.

$\diamond$

Below we follow the notation of Grunschlag [13], distinguishing between the "*input*" and the "*given data*", the information that can be used by the algorithm *"for free"*, that is it does not affect the complexity issues.

Algorithm

Given: Finite groups G_1, G_2, A and the amalgam $G = G_1 *_A G_2$ given via (1.a), (1.b) and (1.c), respectively.

We assume that the Cayley graphs and all the relative Cayley graphs of the free factors are given.

Input: A finite set $\{g_1, \cdots, g_n\} \subseteq G$.

Output: A finite graph $\Gamma(H)$ with a basepoint v_0 which is a reduced precover of G and the following holds

- $Lab(\Gamma(H), v_0) =_G H$;
- $H = \langle g_1, \cdots, g_n \rangle$;
- a normal word w is in H if and only if there is a loop (at v_0) in $\Gamma(H)$ labelled by the word w.

Notation: Γ_i is the graph obtained after the execution of the i-th step.

Step1: Construct a based set of n loops around a common distinguished vertex v_0, each labelled by a generator of H;

Step2: Iteratively fold edges and cut hairs;

Step3:

`For` each X_i-monochromatic component C of Γ_2 $(i = 1, 2)$ `Do`

`Begin`

pick an edge $e \in E(C)$;

glue a copy of $Cayley(G_i)$ on e via identifying 1_{G_i} with $\iota(e)$ and identifying the two copies of e in $Cayley(G_i)$ and in Γ_2;

`If` necessary `Then` iteratively fold edges;

`End;`

Step4:

`For` each $v \in VB(\Gamma_3)$ `Do`

`If` there are paths p_1 and p_2, with $\iota(p_1) = \iota(p_2) = v$ and

$\tau(p_1) \neq \tau(p_2)$ such that

$$lab(p_i) \in G_i \cap A \ (i = 1, 2) \text{ and } lab(p_1) =_G lab(p_2)$$

`Then` identify $\tau(p_1)$ with $\tau(p_2)$;
`If` necessary `Then` iteratively fold edges;

Step5: Reduce Γ_4 by iteratively removing all *redundant* X_i-monochromatic components C such that

- (C, ϑ) is isomorphic to $Cayley(G_i, K, K \cdot 1)$, where $K \leq A$ and $\vartheta \in VB(C)$;
- $|VB(C)| = [A : K]$;
- one of the following holds
  - $K = \{1\}$ and $v_0 \notin VM_i(C)$;
  - K is a nontrivial subgroup of A and $v_0 \notin V(C)$.

Let Γ be the resulting graph;

`If` $VB(\Gamma) = \emptyset$ and (Γ, v_0) is isomorphic to $Cayley(G_i, 1_{G_i})$
`Then` we set $V(\Gamma_5) = \{v_0\}$ and $E(\Gamma_5) = \emptyset$;
`Else` we set $\Gamma_5 = \Gamma$.

Step6:
`If`

- $v_0 \in VM_i(\Gamma_5)$ $(i \in \{1, 2\})$;
- (C, v_0) is isomorphic to $Cayley(G_i, K, K \cdot 1)$, where $L = K \cap A$ is a nontrivial subgroup of A and C is a X_i-monochromatic component of Γ_5 such that $v_0 \in V(C)$;

`Then` glue to Γ_5 a X_j-monochromatic component $(1 \leq i \neq j \leq 2)$ $D = Cayley(G_j, L, L \cdot 1)$ via identifying $L \cdot 1$ with v_0 and identifying the vertices $L \cdot a$ of $Cayley(G_j, L, L \cdot 1)$ with the vertices $v_0 \cdot a$ of C, for all $a \in A \setminus L$.

Denote $\Gamma(H) = \Gamma_6$.

Remark 8.4. The first two steps of the above algorithm correspond precisely to the Stallings' folding algorithm for finitely generated subgroups of free groups (see [35, 25, 17]). This allows one to refer to our algorithm as the *generalized Stallings' (folding) algorithm* for finitely generated subgroups of amalgams of finite groups.

By the results of [35, 25, 17], the graph Γ_2 is finite, well-labelled with $X^{\pm}$, has no hairs and $Lab_{F(X)}(\Gamma_2, v_0) = H$, where $Lab_{F(X)}(\Gamma_2, v_0)$ is the image of $lab(Loop(\Gamma_2, v_0))$ in the free group $F(X)$.

$\diamond$

Example 8.5. Let $G = gp\langle x, y | x^4, y^6, x^2 = y^3 \rangle$.

Let H_1 and H_2 be finitely generated subgroups of G such that

$$H_1 = \langle xy \rangle \text{ and } H_2 = \langle xy^2, yxyx \rangle.$$

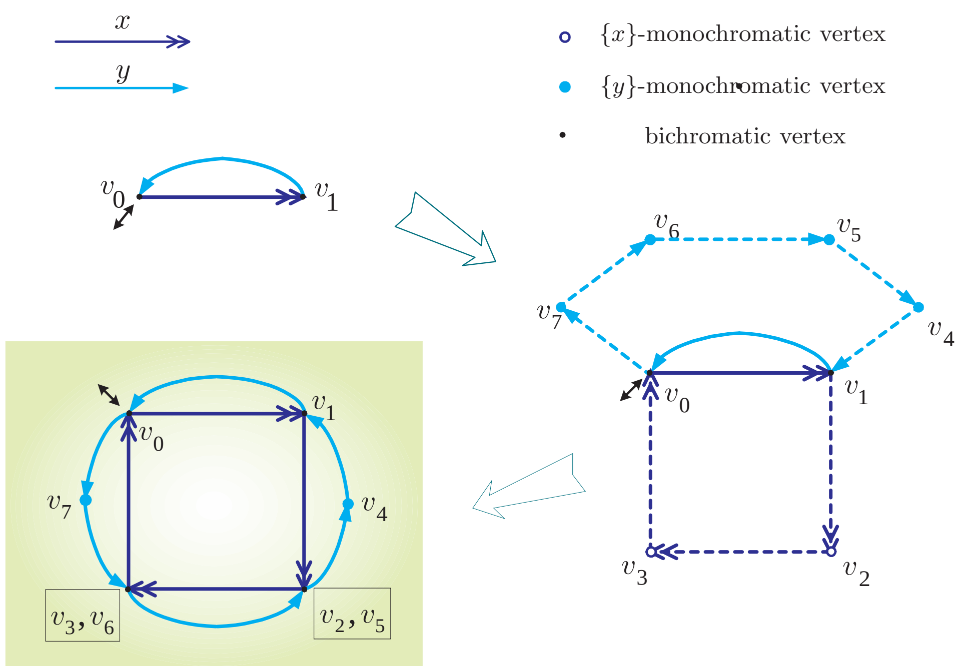

FIGURE 13. The construction of $\Gamma(H_1)$.

The construction of $\Gamma(H_1)$ and $\Gamma(H_2)$ by the algorithm presented above is illustrated on Figure 13 and Figure 14. $\diamond$

Lemma 8.6. *The algorithm terminates and constructs the graph* $(\Gamma(H), v_0)$ *which is a finite reduced precover of* G *with* $Lab(\Gamma(H), v_0) = H$*.*

Proof. By Remark 8.4, the first two steps of the algorithm terminates and construct the finite graph Γ_2. Since G_1 and G_2 are finite groups, $Cayley(G_1)$ and $Cayley(G_2)$ are finite graphs. Therefore, by the construction, all the intermediate graphs Γ_i $(3 \leq i \leq 6)$ are finite. Moreover they are constructed by a finite sequence of iterations. Thus the resulting graph $\Gamma(H)$ is finite.

By Remark 8.4 and by Lemma 3.7, $Lab(\Gamma_2, v_0) = Lab_{F(X)}(\Gamma_2, v_0) = H$.

Applying to each of the intermediate graphs Γ_i $(3 \leq i \leq 6)$ the appropriate lemma from Lemmas 3.7, A.1 (see Appendix), 3.10, 6.17 and 6.21, we get

$$Lab(\Gamma_6, v_0) = Lab(\Gamma_5, v_0) = Lab(\Gamma_4, v_0) = Lab(\Gamma_3, v_0) = Lab(\Gamma_2, v_0) = H.$$

Thus $Lab(\Gamma(H), v_0) = H$.

Graphs Γ_3 and Γ_4 are well-labelled with $X^{\pm}$, due to the folding operations, by [35].

Γ_3 has no hairs. Indeed, since the graphs Γ_2 and $Cayley(G_i)$ $(i \in \{1, 2\})$ have no hairs, the intermediate graph of the third step obtained after the gluing operations has no hairs. Moreover, the graphs Γ_2 and $Cayley(G_i)$ $(i \in \{1, 2\})$ are well-labelled. Thus the only possible foldings in the intermediate graph are between edges of a X_i-monochromatic component C of Γ_2 and edges of the copy of $Cayley(G_i)$ $(i \in \{1, 2\})$ glued to C along the common

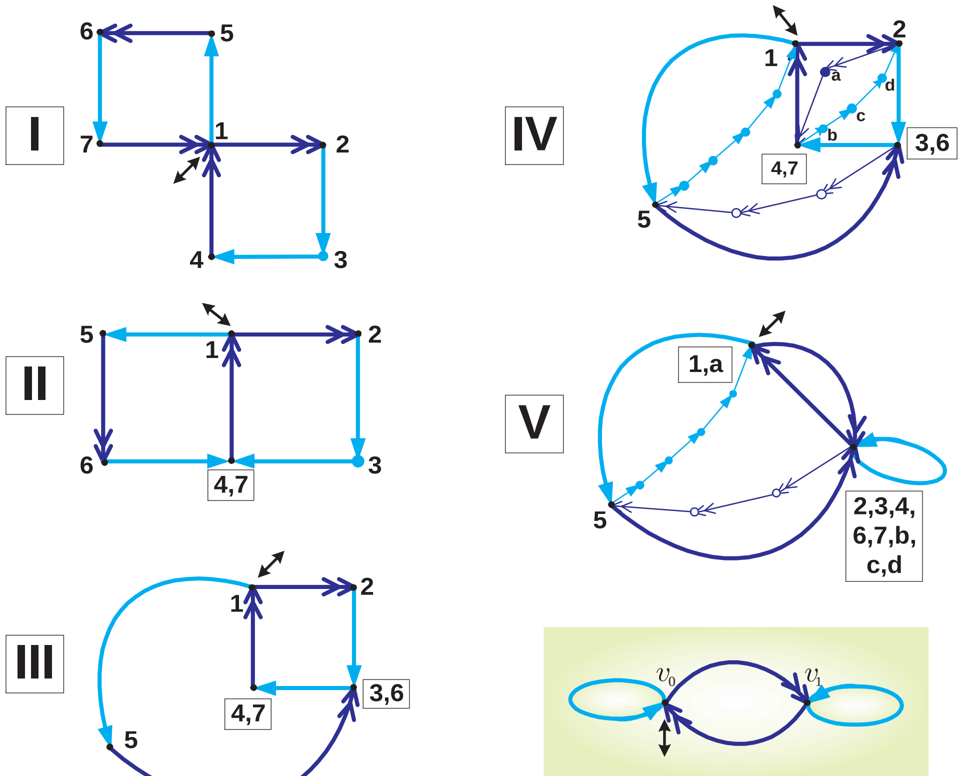

FIGURE 14. The construction of $\Gamma(H_2)$.

edge e. Therefore the terminal vertices of the resulting edges have degree greater than 1.

Since foldings keep $X_i^{\pm}$-saturated vertices $X_i^{\pm}$-saturated and keep closed paths closed, the image of a copy of $Cayley(G_i)$ $(i \in \{1,2\})$ in Γ_3 remains G_i-based and $X_i^{\pm}$-saturated. Thus by Lemma 4.1, it is a cover of G_i.

Let C be a X_i-monochromatic component of Γ_2 from the definition of the third step, that is $e \in E(C)$. Let C' be its image in Γ_3. Then $C' \subseteq S$, where S is an image of a copy of the Cayley graph $Cayley(G_i)$ $(i \in \{1,2\})$ in Γ_3. Indeed, let $v' \in V(C')$ be the image of the vertex $v \in V(C)$. Hence, since C is connected, there exist a path q in C such that $\iota(q) = \iota(e)$ and $\tau(q) = v$. Thus $lab(q) \in (X_i^{\pm})^*$. Since the graph operations of the third step can be viewed as graph morphisms, they preserves labels and "commutes" with endpoints. Thus the image q' of the path q in C' satisfies $\iota(q') = \vartheta$, $\tau(q') = v'$ and $lab(q') \equiv lab(q)$, where ϑ is the "common" image in Γ_3 of the vertices $\iota(e)$ of Γ_2 and 1_{G_i} of $Cayley(G_i)$.

On the other hand, since $Cayley(G_i)$ is $X_i^{\pm}$ saturated, there exists a path γ in $Cayley(G_i)$ with $\iota(\gamma) = 1_{G_i}$ and $lab(\gamma) \equiv lab(q)$. Then there is a path γ' in S with $\iota(\gamma') = \vartheta$ and $lab(\gamma') \equiv lab(\gamma) \equiv lab(q)$. Since Γ_3 is well-labelled, we have $q' = \gamma'$. Hence $V(C') \subseteq V(S)$. Thus $C' \subseteq S$.

Therefore all X_i-monochromatic components of Γ_3 are covers of G_i ($i \in \{1,2\}$).

Let $v \in VB(\Gamma_3)$ and let p_1 and p_2 be paths in Γ_3 such that

- $\iota(p_1) = \iota(p_2) = v$;
- $\tau(p_1) \neq \tau(p_2)$;
- $lab(p_i) \in G_i \cap A$ ($i = 1, 2$);
- $lab(p_1) =_G lab(p_2)$.

Then $v \in VB(\Gamma_3)$ and $deg(v) \geq 2$.

Let ν be a vertex which is the result of the identification of the vertices $\tau(p_1)$ and $\tau(p_2)$ of Γ_3. If $\tau(p_1)$ and $\tau(p_2)$ are monochromatic vertices of Γ_3 of different colors, then no foldings are possible at ν and $deg(\nu) \geq 2$.

Otherwise at least one of them is bichromatic in Γ_3. Then ν is a bichromatic vertex of Γ_4 and foldings are possible at ν. However foldings keep bichromatic vertices bichromatic. Thus $\nu \in VB(\Gamma_4)$ and $deg(\nu) \geq 2$.

Therefore, since Γ_3 has no hairs, Γ_4 has no hairs as well. By Remarks 3.9 and 6.5, each X_i-monochromatic component of Γ_4 is a cover of G_i ($i \in \{1,2\}$). By the construction, Γ_4 is compatible. Hence, by Lemma 6.8, the graph Γ_4 is a precover of G.

By Lemma 6.17, Γ_5 is a precover of G as well. Since, by the construction, Γ_5 has no redundant monochromatic components, Lemma 6.21 implies that $\Gamma(H) = \Gamma_6$ is a reduced precover of G.

$\diamond$

Now we sketch the complexity analysis of the above algorithm.

Lemma 8.7 (Complexity)**.** *Let m be the sum of the lengths of words $g_1, \dots g_n$. Then the algorithm computes $(\Gamma(H), v_0)$ in time $O(m^2)$.*

Proof. As is well known, see [5], the construction of the bouquet can be done in time proportional to m, foldings can be implemented in time proportional to m^2 and cutting hairs can be done in time proportional to m. Therefore the first two steps of the algorithm are completed in time $O(m^2)$, while the graph Γ_2 satisfies: $|E(\Gamma_2)| \leq m$ and $|V(\Gamma_2)| \leq m$.

Given Cayley graphs of both free factors G_1 and G_2, the gluing operations of the third step take time proportional to m, because we just identify one edge of each monochromatic component of Γ_2 (whose detecting takes $|E(\Gamma_2)|$) with the corresponding edge of the graph $Cayley(G_i)$, $i \in \{1,2\}$.

Note that $|V(\Gamma_3)| = k_1 \cdot |G_1| + k_2 \cdot |G_2|$, where k_i, $i \in \{1,2\}$, is a number of X_i-monochromatic components of Γ_2. Since the information about the factors G_1 and G_2 is given, that is it is not a part of the input, and since $k_1 + k_2 \leq m$, we conclude that the number $|V(\Gamma_3)|$ is proportional to m. Similarly, $|E(\Gamma_3)|$ is proportional to m as well.

The detecting of bichromatic vertices of Γ_3 takes time proportional to $|V(\Gamma_3)|$, that is it takes time proportional to m. By the proof of Lemma 8.6 (the proof of the fourth step), there are at most $|A|$ identifications for each

bichromatic vertex of Γ_3. Thus the identifications of the fourth step take at most $|VB(\Gamma_3)| \cdot |A|$. However, the description of the third step implies that

$$|VB(\Gamma_3)| = |VB(\Gamma_2)| \leq |V(\Gamma_2)|.$$

Since the number of vertices of the intermediate graph of the fourth step obtained just after the above identifications is not greater than $|V(\Gamma_3)|$, the foldings operations applied to this graph can be implemented in time proportional to $|V(\Gamma_3)|^2$, by [5]. Since $|V(\Gamma_3)|$ is proportional to m, it takes time proportional to m^2. Thus, summarizing the analysis of the fourth step, we see that its implementation takes $O(m^2)$.

The indication of connected monochromatic components of Γ_4 takes time proportional to $|E(\Gamma_4)|$. Since $|E(\Gamma_4)| \leq |E(\Gamma_3)|$ and $|E(\Gamma_3)|$ is proportional to m, this procedure takes time proportional to m.

By the proof of Lemma 8.6, the graph Γ_4 is a precover of G, hence its X_i-monochromatic components are covers of G_i for all $i \in \{1, 2\}$. Since the information about the factors G_1 and G_2 is given, that is it is not a part of the input, the verifications concerning monochromatic components of Γ_4 take $O(1)$.

Since in the worst case the monochromatic component of Γ_4 that has to be deleted via the fifth step might appear at the end of the verification process, while it induces a series of deletions, the fifth step can be completed in time proportional to $|E(\Gamma_4)|$, that is in $O(m^2)$.

The last step of the algorithm takes at most $|A|$, that is constant according to our assumption (it is a part of the 'given information").

Summarizing the above description of the steps complexity, we conclude that the algorithm constructs the resulting graph $\Gamma(H)$ in time $O(m^2)$.

$\diamond$

Remark 8.8. Note that if the group presentations of the free factors G_1 and G_2, as well as the monomorphisms between the amalgamated subgroup A and the free factors are a part of the input (the *uniform version* of the algorithm) then we have to build the groups G_1 and G_2 (that is to construct their Cayley graphs and relative Cayley graphs).

Since we assume that the groups G_1 and G_2 are finite, the Todd-Coxeter algorithm and the Knuth Bendix algorithm are suitable [23, 34, 36] for these purposes. Then the complexity of the construction depends on the group presentation of G_1 and G_2 we have: it could be even exponential in the size of the presentation. Therefore the generalized Stallings' folding algorithm with these additional constructions could take time exponential in the size of the input.

$\diamond$

Theorem 8.9. *Let Y be a finite subset of G and let $H = \langle Y \rangle$ be a finitely generated subgroup of G. Then the resulting graph $(\Gamma(H), v_0)$ constructed by the generalized Stallings' folding algorithm is the normal core of $Cayley(G, H)$.*

Proof. The generalized Stallings' folding algorithm constructs a graph $(\Gamma(H), v_0)$, which is a finite reduced precover of G with $Lab(\Gamma(H), v_0) = H$, by Lemma 8.6. Hence, by Theorem 7.2, $(\Gamma(H), v_0)$ is isomorphic to $(\Delta, H \cdot 1)$, the normal core of $Cayley(G, H)$. Since this isomorphism is unique, by Remark 3.6, the graph $(\Gamma(H), v_0)$ can be identified with the normal core of $Cayley(G, H)$. $\diamond$

Remark 8.10 (Canonicity and Constructibility)**.** Theorem 8.9 implies that the normal core of a relative Cayley graph is constructible.

Since, by Definition 5.2 and Remark 5.3, the normal core of $Cayley(G, H)$ depends on H itself and not on the set of subgroup generators, Theorem 8.9 implies that the graph $(\Gamma(H), v_0)$ is canonically associated with H. $\diamond$

As an immediate consequence of Theorem 8.9 we get the following corollary, which provide a solution for the membership problem for finitely generated subgroups of amalgams of finite groups. We discuss it in the next section.

Corollary 8.11. *A normal word g is in H if and only if it labels a closed path in $\Gamma(H)$ starting at v_0.*

Proof. A normal word g is in H if and only if it labels a normal path in the normal core of $Cayley(G, H)$ closed at $H \cdot 1$. Since, by Theorem 8.9, $(\Gamma(H), v_0)$ constructed by the generalized Stallings' folding algorithm is the normal core of $Cayley(G, H)$, we obtain the desired conclusion.

$\diamond$

9. The Membership Problem

The *membership problem* (or the *generalized word problem*) for a subgroup of a given group asks to decide whether a word in the generators of the group is an element of the given subgroup.

As is well known ([1]), the membership problem for finitely generated subgroups is solvable in amalgams of finite groups. Different types of solutions can be found in [8, 15, 13] and other sources.

Below we introduce a solution of the membership problem for finitely generated subgroups of amalgams of finite groups which employs subgroup graphs (normal cores) constructed by the generalized Stallings' foldings algorithm, presented in Section 8.

Corollary 9.1. *Let $g, h_1, \ldots h_n \in G$. Then there exists an algorithm which decides whether or not g belongs to the subgroup $H = \langle h_1, \ldots, h_n \rangle$ of G.*

Proof. First we construct the graph $\Gamma(H)$, using the algorithm from Section 8. By Corollary 8.11, $g \in H$ if and only if there is a normal path p in $\Gamma(H)$ closed at the basepoint v_0 such that $lab(p) =_G g$. That is the word $lab(p)$ is a normal form of the word g.

Thus in order to decide if $g \in H$ we have to begin with a calculation of a normal form $\overline{g}$ of the given word g. If g is a normal word then we just skip

the calculation and put $\overline{g} \equiv g$. Otherwise we use a well-known rewriting procedure [23] to find $\overline{g}$. This usage is possible because the membership problem for the amalgamated subgroup A is solvable in the free factors G_1 and G_2 (indeed, they are finite groups).

Now we have to verify if there exists a path p in $\Gamma(H)$ closed at the basepoint v_0 such that $lab(p) \equiv \overline{g}$. It can be done as follows. We start at the vertex v_0 and try to read the word $\overline{g}$ in the graph $\Gamma(H)$. If we become stuck during this process or if we don't return to the vertex v_0 at the end of the word $\overline{g}$, then g is not in H. Otherwise we conclude that $g \in H$.

$\diamond$

Example 9.2. Let H_2 be the subgroup of G from Example 8.5. Then using Figure 14 and the algorithm described in Corollary 9.1, we easily conclude that $xyx \in H_2$, because $v_0 \cdot (xyx) = v_0$ in $\Gamma(H_2)$. But $xy^3x^{-5} \notin H_2$, because $v_0 \cdot (xy^3x^{-5}) \neq v_0$. $\diamond$

The algorithm presented with the proof of Corollary 9.1 provides a solution for the *membership problem* for finitely generated subgroups of amalgams of finite groups with the following description.

GIVEN: Finite groups G_1, G_2, A and the amalgam $G = G_1 *_A G_2$ given via (1.a), (1.b) and (1.c), respectively.

We assume that the Cayley graphs and all the relative Cayley graphs of the free factors are given.

INPUT: Words $g, h_1, \ldots, h_n \in G$.

DECIDE: Whether or not g belongs to the subgroup $H = \langle h_1, \ldots, h_n \rangle$.

Complexity. Let m be the sum of the lengths of the words $h_1, \ldots h_n$. By Lemma 8.7, the algorithm from Section 8 computes $(\Gamma(H), v_0)$ in time $O(m^2)$. The verification of the normality of the word g is proportional to $|g|$ and the computation of its normal form takes time $O(|g|^2)$. To read a normal word in the graph $(\Gamma(H), v_0)$ in the way, explained in the proof of Corollary 9.1, takes time equal to the length of the word. Therefore the complexity of the algorithm is $O(m^2 + |g|^2)$.

If in the above description the input is changed to:

INPUT: Words $h_1, \ldots, h_n \in G$ and a normal word $g \in G$.

then the complexity of the algorithm will be $O(m^2 + |g|)$.

In some papers, the following slightly different description of the *membership problem* can be found.

GIVEN: Finite groups G_1, G_2, A and the amalgam $G = G_1 *_A G_2$ given via (1.a), (1.b) and (1.c), respectively.

We assume that the Cayley graphs and all the relative Cayley graphs of the free factors are given.

The subgroup $H = \langle h_1, \ldots, h_n \rangle$ of G.

INPUT: A normal word $g \in G$.

DECIDE: Whether or not g belongs to the subgroup H.

In this context the subgroup H is given, that is $(\Gamma(H), v_0)$ is constructed and can be used for free. Therefore the complexity of this algorithm is linear in the length of the word g, because we simply have to read it in the graph $(\Gamma(H), v_0)$ which takes time equal to $|g|$.

Another variation of the *membership problem* is the *uniform membership problem*, when the presentation of the group G is a part of the input.

GIVEN: -
INPUT: Finite groups G_1, G_2, A and the amalgam $G = G_1 *_A G_2$ given via $(1.a)$, $(1.b)$ and $(1.c)$, respectively.
Words $g, h_1, \ldots, h_n \in G$.
DECIDE: Whether or not g belongs to the subgroup $H = \langle h_1, \ldots, h_n \rangle$.

Complexity. The algorithm given along with the proof of Corollary 9.1 provide a solution for the above problem. However now the complexity of the construction of $(\Gamma(H), v_0)$ might be exponential in the size of the presentation $(1.a)$-$(1.c)$, by Remark 8.8. Therefore the complexity of the algorithm might be exponential in the size of the input.

Appendix A.

When constructing graphs for subgroups of non free groups, nontrivial relations of these groups have to be taken into account. Roughly speaking, they have to be "sewed" somehow on subgroup graphs. This gives rise to a *gluing* operation on the graph. Moreover, if one is interested to construct a precover of an amalgam then the gluing operation of copies of Cayley graphs of the free factors to the graph have to be defined.

Let Γ be a graph well-labelled with $X^{\pm}$. Let $e \in E(\Gamma)$ such that $lab(e) \equiv w \in X_j$ $(j \in \{1, 2\})$.

Let Γ' be the graph constructed by taking the disjoint union of the graphs Γ and $Cayley(G_j)$ via the identification of the edge $e \in E(\Gamma)$ with the edge $f \in E(Cayley(G_j))$ such that $\iota(f) = 1_{G_j}$ and $lab(f) \equiv lab(e)$.

More precisely,

$$V(\Gamma') = (V(\Gamma) \setminus \{\iota(e), \tau(e)\}) \cup (V(Cayley(G_j)) \setminus \{\iota(f), \tau(f)\}) \cup \{v_1, v_2\}.$$

$$E(\Gamma') = (E(\Gamma) \setminus \{e\}) \cup (E(Cayley(G_j)) \setminus \{f\}) \cup \{l\}.$$

The endpoints and arrows for the edges of Γ' are defined in a natural way.

$$\iota_{\Gamma'}(\xi) = \begin{cases} \iota_{\Gamma}(\xi), & \text{if } \xi \in E(\Gamma) \setminus \{e\}; \\ \iota_{Cayley(G_j)}(\xi), & \text{if } \xi \in E(Cayley(G_j)) \setminus \{f\}; \\ v_1, & \text{if } \xi = l; \\ v_2, & \text{if } \xi = \overline{l}. \end{cases}$$

We define labels on the edges of Γ' as follows:

$$lab_{\Gamma'}(\xi) \equiv \begin{cases} lab_{\Gamma}(\xi), & \text{if } \xi \in E(\Gamma) \setminus \{e\}; \\ lab_{Cayley(G_j)}(\xi), & \text{if } \xi \in E(Cayley(G_j)) \setminus \{f\}; \\ w, & \text{if } \xi = l. \end{cases}$$

We say that Γ' is obtained from Γ by *gluing a copy of* $Cayley(G_j)$ *along the edge* e *of* Γ.

Lemma A.1. *Let* Γ' *be the graph obtained from the well-labelled graph* Γ *gluing a copy of* $Cayley(G_j)$ *along the edge* e *of* Γ.

Then $Lab(\Gamma, v_0) = Lab(\Gamma', v_0')$, *where* v_0 *is the basepoint of* Γ *and* v_0' *is the (corresponding) basepoint of* Γ'.

Proof. Since the graph $Cayley(G_j)$ is $X_j^{\pm}$-saturated, there exists an edge $f \in E(Cayley(G_j))$ such that $\iota(f) = 1_{G_j}$ and $lab(f) \equiv lab(e)$. Thus the construction of Γ' is possible.

By the construction, Γ and $Cayley(G_j)$ embed in Γ'. Hence $(\Gamma, v_0) \subseteq (\Gamma', v_0')$, thus $Loop(\Gamma, v_0) \subseteq Loop(\Gamma', v_0')$. Therefore $Lab(\Gamma, v_0) \subseteq Lab(\Gamma', v_0')$.

Let $u \in Lab(\Gamma', v_0')$. Then there is $t' \in Loop(\Gamma', v_0')$ such that $lab(t') =_G u$. If $t' \subseteq \Gamma$ then $lab(t') =_G u \in Lab(\Gamma, v_0)$. Otherwise there is a decomposition

$$t' = t_1' q_1 t_2' q_2 \dots q_{k-1} t_k',$$

such that $\iota(t_1') = \tau(t_k') = v_0'$, and for all $1 \leq i \leq k$, $t_i' \subseteq \Gamma$ and q_i is a path in Γ' which doesn't exist in Γ.

Thus for all $1 \leq i \leq k$, q_i is in $Cayley(G_j)$ and $\iota(q_i), \tau(q_i) \in \{v_1, v_2\}$, where v_1 and v_2 are images of the vertices $\iota(e)$ and $\tau(e)$ of Γ in Γ', respectively. Then either $\iota(q_i) = \tau(q_i)$ or $\iota(q_i) \neq \tau(q_i)$. In the first case q_i is a closed path in $Cayley(G_j)$, hence $lab(q_i) =_{G_j} 1$. In the second case either $\iota(q_i) = \iota(f)$, $\tau(q_i) = \tau(f)$ or $\iota(q_i) = \tau(f)$, $\tau(q_i) = \iota(f)$. Therefore $lab(q_i) =_{G_j} lab(f)$ or $lab(q_i) =_{G_j} (lab(f))^{-1}$, respectively.

Let $t = t_1' q_1' t_2' q_2' \dots q_k' t_k'$ be a path in Γ such that for all $1 \leq i \leq k$,

$$q_i' = \begin{cases} \emptyset, & \iota(q_i) = \tau(q_i); \\ e, & lab(q_i) =_{G_j} lab(f); \\ \overline{e}, & lab(q_i) =_{G_j} (lab(f))^{-1}. \end{cases}$$

By $q_i' = \emptyset$ we mean that q_i' is the empty path, that is $lab(q_i') =_{G_j} 1$, with the desired initial-terminal vertex $\iota(g_i') = \tau(q_i') = \tau(t_i') = \iota(t_{i+1}')$.

Since $lab(e) \equiv lab(f)$,

$$lab(q_i') = \begin{cases} 1, & \iota(q_i) = \tau(q_i); \\ lab(f), & lab(q_i) =_{G_j} lab(f); \\ (lab(f))^{-1}, & lab(q_i) =_{G_j} (lab(f))^{-1}. \end{cases}$$

Thus $lab(q_i') =_{G_j} lab(q_i)$. Therefore

$$\begin{aligned} lab(t) &\equiv lab(t_1')lab(q_1')lab(t_2')lab(q_2') \dots lab(q_k')lab(t_k') \\ &=_{G_j} lab(t_1')lab(q_1)lab(t_2')lab(q_2) \dots lab(q_k)lab(t_k') \\ &\equiv lab(t'). \end{aligned}$$

Since $lab(t') \in Lab(\Gamma, v_0)$, we have $Lab(\Gamma) = Lab(\Gamma')$.

$\diamond$

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